

# RIGHT AND LEFT JOINT SYSTEM REPRESENTATION OF A RATIONAL MATRIX FUNCTION IN GENERAL POSITION (SYSTEM REPRESENTATION THEORY FOR DUMMIES)

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For a rational  $k \times k$  matrix function  $R$  of one variable in general position, the matrix functions  $R(x) \cdot R^{-1}(y)$  and  $R^{-1}(x) \cdot R(y)$  of two variables are considered. For these matrix functions of two variables, the representations which are analogous to the system representation (or realization) of a rational matrix function of one variable are constructed. This representation of the function  $R(x) \cdot R^{-1}(y)$  (of the function  $R^{-1}(x) \cdot R(y)$ ) is said to be the joint right (respectively the joint left) system representation of the matrix functions  $R, R^{-1}$ . In these representations there appear diagonal  $n \times n$  matrices,  $A_{\mathcal{P}} = \text{diag}(\lambda_1, \dots, \lambda_n)$  (called the pole matrix for  $R$ ) and  $A_{\mathcal{N}} = \text{diag}(\mu_1, \dots, \mu_n)$  (called the zero matrix for  $R$ ), where  $\lambda_1, \dots, \lambda_n$  are poles of  $R$ ,  $\mu_1, \dots, \mu_n$  are poles of  $R^{-1}$ ; and  $k \times n$  matrices  $F_{\mathcal{P}}$  and  $F_{\mathcal{N}}$  (called the left pole and zero semi-residual matrices) and  $n \times k$  matrices  $G_{\mathcal{P}}$  and  $G_{\mathcal{N}}$  (called the right pole and zero semi-residual matrices) which can be introduced from the additive decompositions  $R(z) = R(\infty) + F_{\mathcal{P}}(zI - A_{\mathcal{P}})^{-1}G_{\mathcal{P}}$ ,  $R^{-1}(z) = R(\infty)^{-1} + F_{\mathcal{N}}(zI - A_{\mathcal{N}})^{-1}G_{\mathcal{N}}$ . The right joint system representation has the form  $R(x) \cdot R^{-1}(y) = I + (x - y)F_{\mathcal{P}}(xI - A_{\mathcal{P}})^{-1}(S^r)^{-1}(yI - A_{\mathcal{N}})^{-1}G_{\mathcal{N}}$ , the left one has the form  $R^{-1}(x) \cdot R(y) = I + (x - y)F_{\mathcal{N}}(xI - A_{\mathcal{N}})^{-1}(S^l)^{-1}(yI - A_{\mathcal{P}})^{-1}G_{\mathcal{P}}$ . The  $n \times n$  matrices  $S^r$  and  $S^l$  (the so-called right and left zero-pole coupling matrices for  $R$ ) are solutions of the appropriate Sylvester-Lyapunov equations. These matrices are mutually inverse:  $S^r \cdot S^l = S^l \cdot S^r = I$ .

These results are essentially not new: they could be easily derived from known results on realization of a rational matrix functions (for example, from results by L. Sakhnovich or J. Ball, I. Gohberg, L. Rodman), however the method is new, as well as the emphasis on “the left, the right and their relationships”. The presentation is oriented to a “traditional” analyst. No previous knowledge in realization theory of matrix functions or its ideology is assumed. One of the purposes of this paper is to provide a realization theory background for investigations of the deformation theory of Fuchsian differential system and of rational solutions of the Schlesinger system. As an application we also consider the spectral (Wiener-Hopf) factorization.

The concluding Section 5 contains some historical remarks highlighting the role of M.S. Livšic as the forefather of the system realization theory.

## NOTATIONS.

- $\mathbb{C}$  stands for the complex plane;  $\overline{\mathbb{C}}$  is the extended complex plane:  $\overline{\mathbb{C}} \stackrel{\text{def}}{=} \mathbb{C} \cup \infty$ ;
- $\mathfrak{M}_k$  stands for the set of all  $k \times k$  matrices with complex entries;
- $I$  stands for the unity matrix of the appropriate dimension;
- $\mathcal{R}(\mathfrak{M}_k)$  stands for the set of all rational  $\mathfrak{M}_k$ -valued functions  $R$  with  $\det R(z) \neq 0$ .
- $\mathcal{P}(R)$  stands for the set of all poles of the function  $R$ ,  $\mathcal{N}(R)$  stands for the set of all poles of the function  $R^{-1}$ ;  $\mathcal{P}(R)$  is said to be the pole set of the function  $R$ ,  $\mathcal{N}(R)$  is said to be the zero set of the function  $R$ .

## 0. PREFACE

The problem which we set as a goal in this paper for scalar (i.e. complex valued) functions means *to restore a rational function from its poles and zeros*. The traditional solution of this problem uses products constructed from the poles and zeros of the function. For rational functions *in general position*, this solution can be explained particularly clearly.

Namely, let  $r$  be a rational function in general position<sup>1</sup>, with the pole set  $\mathcal{P}(r)$  and the zero set  $\mathcal{N}(r)$ . These sets  $\mathcal{P}(r)$  and  $\mathcal{N}(r)$  do not intersect (i.e.  $\mathcal{P}(r) \cap \mathcal{N}(r) = \emptyset$ ) and are of the same cardinality:  $\#\mathcal{P}(r) = \#\mathcal{N}(r)$ . The function  $r$  admits the representation

$$r(z) = c \left( \prod_{\mu_l \in \mathcal{N}(r)} (z - \mu_l) \right) \cdot \left( \prod_{\lambda_j \in \mathcal{P}(r)} (z - \lambda_j) \right)^{-1}, \quad (0.1)$$

where  $c = r(\infty)$ . This *multiplicative* representation recovers the function  $r$  from its pole and zero sets and from the value  $r(\infty)$ . Inversely, given two finite non-intersecting sets  $\mathcal{P}$  and  $\mathcal{N}$  ( $\mathcal{P}, \mathcal{N} \in \mathbb{C}$ ,  $\mathcal{P} \cap \mathcal{N} = \emptyset$ ) of the same cardinality and a complex number  $c \neq 0, \infty$ , we define the function  $r$  by the formula (0.1). This function  $r$  is a rational function in general position, the given sets  $\mathcal{P}$  and  $\mathcal{N}$  are its pole and zero sets  $\mathcal{P}(r)$  and  $\mathcal{N}(r)$  and  $c = r(\infty)$ .

However, in view of non-commutativity of the matricial multiplication, the multiplicative representation (0.1) seems to be unsuitable for generalization to **matrix** functions. We present now such a representation of a rational matrix function (in general position) from its poles and zeros which can be generalized to the matricial case. This is the so-called **system representation** of a rational function.

So, let again  $r$  be a rational function in general position, with the pole and zero sets  $\mathcal{P}(r)$  and  $\mathcal{N}(r)$ . We derive its system representation. We assume for simplicity that the function  $r$  is normalized by the condition  $r(\infty) = 1$ . We start from the additive decomposition of the matrix function  $r$ :

$$r(z) = 1 + \sum_{\lambda_q \in \mathcal{P}(r)} \frac{\xi_q}{z - \lambda_q}. \quad (0.2)$$

The condition

$$r(\mu_p) = 0 \quad (\forall \mu_p \in \mathcal{N}(r)) \quad (0.3)$$

leads to the system of linear equations

$$\sum_{\lambda_q \in \mathcal{P}(r)} \frac{\xi_q}{\mu_p - \lambda_q} = -1 \quad (\forall \mu_p \in \mathcal{N}(r)). \quad (0.4)$$

Thus, to restore a rational function in general position from its poles and zeros, we have to solve the linear system (0.4) with respect to  $\xi_q$  and then to substitute these  $\xi_q$  into (0.2). Since  $\#\mathcal{P}(r) = \#\mathcal{N}(r) \stackrel{\text{def}}{=} n$ , the matrix  $S$  of the system (0.4) is square:

$$S = \|s_{p,q}\|_{1 \leq p, q \leq n}, \quad s_{p,q} = \frac{1}{\mu_p - \lambda_q}. \quad (0.5)$$

The system (0.4) is uniquely solvable: its determinant (which is known as the *Cauchy determinant*) can be calculated explicitly (see, for example, [PS], Pt.VII: §1, no. **3**). From this explicit expression for the determinant it is evident that  $\det S \neq 0$ .

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<sup>1</sup>This means that all poles and zeros of the function  $r$  are simple and  $r(\infty) \neq 0, \infty$ .

We can formulate this method of restoring of the function  $r$  from  $\mathcal{P}(r)$  and  $\mathcal{N}(r)$  in the matricial form. Let  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  be the diagonal matrices constructed from  $\mathcal{P}(r)$  and  $\mathcal{N}(r)$ :

$$A_{\mathcal{P}} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n), \quad A_{\mathcal{N}} = \text{diag}(\mu_1, \mu_2, \dots, \mu_n). \quad (0.6)$$

Let  $e$  be the  $n$ -row (i.e.  $1 \times n$  matrix):

$$e = [1, 1, \dots, 1]. \quad (0.7)$$

As usual, by  $e^*$  we denote the Hermitian conjugate to  $e$ :  $e^*$  is a  $n$ -column (i.e.  $n \times 1$  matrix). The representation (0.2) can be put down in the form<sup>2</sup>

$$r(z) = 1 + e(zI - A_{\mathcal{P}})^{-1}[\xi_1, \xi_2, \dots, \xi_n]^T. \quad (0.8)$$

The system (0.4) can be presented in matricial form:  $S[\xi_1, \xi_2, \dots, \xi_n]^T = -e^*$ . Thus,

$$[\xi_1, \xi_2, \dots, \xi_n]^T = -S^{-1}e^*.$$

Substituting this expression for  $[\xi_1, \xi_2, \dots, \xi_n]^T$  into (0.8), we come to the representation

$$r(z) = 1 - e(zI - A_{\mathcal{P}})^{-1}S^{-1}e^*. \quad (0.9)$$

In the same way we can obtain the representation for the function  $r^{-1}$ . Starting from the additive representation

$$r^{-1}(z) = 1 + \sum_{\mu_p \in \mathcal{N}(r)} \frac{\eta_p}{z - \mu_p}, \quad (0.10)$$

we come to the linear system (with respect to  $\eta_p$ )

$$\sum_{\mu_p \in \mathcal{N}(r)} \frac{\eta_p}{\mu_p - \lambda_q} = 1 \quad (\forall \lambda_q \in \mathcal{P}(r)),$$

which can be put down in matricial form:

$$[\eta_1, \eta_2, \dots, \eta_n]S = e,$$

or

$$[\eta_1, \eta_2, \dots, \eta_n] = eS^{-1},$$

Thus,

$$r^{-1}(z) = 1 + eS^{-1}(zI - A_{\mathcal{N}})^{-1}e^*. \quad (0.11)$$

Here the matrix  $S$  is the same as in (0.5) !

The matrices  $S$ ,  $A_{\mathcal{P}}$ ,  $A_{\mathcal{N}}$  are involved in the equality

$$A_{\mathcal{N}}S - SA_{\mathcal{P}} = e^*e, \quad (0.12)$$

which can be directly obtained from (0.5), (0.6), (0.7). The equality (0.12) and its generalization are of fundamental importance in the elaborated theory.

The equality of the form  $AX - XB = C$  (where  $A$ ,  $B$ ,  $C$  are square matrices) is known as the Sylvester-Lyapunov equality (or as the Sylvester-Lyapunov equation, if it is considered as an equation with respect to  $X$ ).

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<sup>2</sup>As usual, for the matrix  $M$ ,  $M^T$  denotes the transpose one.

Multiplying the right hand sides of the representations (0.9) and (0.11) term by term, we obtain the equality

$$r(x)r^{-1}(y) = 1 + (x - y)e(xI - A_{\mathcal{P}})^{-1}S^{-1}(yI - A_{\mathcal{N}})^{-1}e^* \quad (0.13)$$

after some calculations. These calculations are based on the Sylvester-Lyapunov equality (0.12). The representations (0.9) and (0.11) are exactly what we need. They are said to be the **system representations** of the function  $\mathbf{r}$  and  $\mathbf{r}^{-1}$  respectively. The representation (0.13) is said to be the **joint system representation** of the pair of (mutually inverses) functions  $\mathbf{r}$  and  $\mathbf{r}^{-1}$ .

We obtained the joint system realization (0.13) from the system representations (0.9) and (0.11). In its turn, the representations (0.9) and (0.11) are contained in (0.13): (0.9) is (0.13) for  $y = \infty$ , (0.11) is (0.13) for  $x = \infty$ .

Now we derive the joint system representation (0.13) in a different way. Let  $k$  be the function

$$k(x, y) = \frac{r(x)r^{-1}(y) - 1}{x - y} \quad (0.14)$$

of two variables. Fixing  $y \notin (\mathcal{P}(r) \cup \mathcal{N}(r) \cup \infty)$ , we consider  $k$  as a function of the variable  $x$ . This function is rational (with respect to  $x$ ), with simple poles located at the points  $\lambda_p \in \mathcal{P}(r)$ , with the residues  $\rho_{\lambda_p}(y)$ :

$$\rho_{\lambda_p}(y) = \frac{1}{(r^{-1})'(\lambda_p)} \cdot \frac{r^{-1}(y)}{\lambda_p - y}. \quad (0.15)$$

It is clear, that  $k(\infty, y) = 0$  and that  $k(x, y)$  is holomorphic for  $x = y$ . Thus,  $k(x, y)$  admits the simple fraction expansion

$$k(x, y) = \sum_{1 \leq p \leq n} \frac{1}{x - \lambda_p} \cdot \rho_{\lambda_p}(y). \quad (0.16)$$

In its turn, the residue  $\rho_{\lambda_p}(y)$ , considered as a function of  $y$ , is a rational one. Since  $r^{-1}(\lambda_p) = 0$ , the point  $\lambda_p$  is not a pole of the function  $\rho_{\lambda_p}$ , and the points  $\mu_q \in \mathcal{N}(r^{-1})$  are the only poles of the function  $\rho_{\lambda_p}$ . All these poles are simple, with the residues  $h_{p,q} = \text{res } \rho_{\lambda_p}(y) |_{y=\mu_q}$ :

$$h_{p,q} = \frac{1}{(r^{-1})'(\lambda_p)} \cdot \frac{1}{\lambda_p - \mu_q} \cdot \frac{1}{r'(\mu_q)}. \quad (0.17)$$

As  $\rho_{\lambda_p}(\infty) = 0$ , the function  $\rho_{\lambda_p}$  admits the simple fraction expansion

$$\rho_{\lambda_p}(y) = \sum_{1 \leq q \leq n} h_{p,q} \frac{1}{y - \mu_q}. \quad (0.18)$$

Substituting (0.18) into (0.16), we come to the formula

$$k(x, y) = \sum_{1 \leq p, q \leq n} \frac{1}{x - \lambda_p} \cdot h_{p,q} \cdot \frac{1}{y - \mu_q}, \quad (0.19)$$

where  $h_{p,q}$  are defined by (0.17). The last formula can be presented in matricial form. Let us organize the numbers  $h_{p,q}$  into the matrix

$$H = \|h_{p,q}\|_{1 \leq p, q \leq n}. \quad (0.20)$$

Thus,

$$k(x, y) = e(xI - A_{\mathcal{P}})^{-1}H(yI - A_{\mathcal{N}})^{-1}e^*. \quad (0.21)$$

and

$$r(x)r^{-1}(y) = 1 + e(xI - A_{\mathcal{P}})^{-1}H(yI - A_{\mathcal{N}})^{-1}e^*. \quad (0.22)$$

The comparison of the formulas (0.13) and (0.22) suggests us that  $H = S^{-1}$ . The equality

$$HS = I \quad (0.23)$$

may be verified starting from the formula

$$r^{-1}(x) = 1 + \sum_{1 \leq q \leq n} \frac{1}{x - \mu_q} \cdot \frac{1}{r'(\mu_q)}. \quad (0.24)$$

The matrix identity (0.23) is equivalent to the system of  $n^2$  scalar identities

$$\sum_{1 \leq q \leq n} h_{p,q} s_{q,t} = 0 \quad (1 \leq p, t \leq n, p \neq t) \quad (0.25)$$

and

$$\sum_{1 \leq q \leq n} h_{p,q} s_{q,p} = 1 \quad (1 \leq p \leq n). \quad (0.26)$$

From (0.24) (and of course, from (0.5) and (0.17)) it follows that<sup>3</sup>

$$\begin{aligned} \sum_{1 \leq q \leq n} h_{p,q} s_{q,t} &= \sum_{1 \leq q \leq n} \frac{1}{\lambda_p - \mu_q} \cdot \frac{1}{\mu_q - \lambda_t} \cdot \frac{1}{(r^{-1})'(\lambda_p)} \cdot \frac{1}{r'(\mu_q)} = \\ &= \frac{1}{(r^{-1})'(\lambda_p)(\lambda_p - \lambda_t)} \left( r^{-1}(\lambda_p) - r^{-1}(\lambda_t) \right) = 0 \quad \text{for } p \neq t, \end{aligned} \quad (0.27)$$

since  $r^{-1}(\lambda_k) = 0$  for all  $\lambda_k \in \mathcal{P}(r) (= \mathcal{N}(r^{-1}))$ . Thus, (0.25) is verified. Analogously, the equality (0.26) may be verified using the formula

$$(r^{-1})'(x) = - \sum_{1 \leq q \leq n} \frac{1}{(x - \mu_q)^2} \cdot \frac{1}{r'(\mu_q)}. \quad (0.28)$$

Thus, the equality (0.23) holds. Finally, the formula (0.22) can be presented in the form (0.13), with  $S$  of the form (0.5).

**REMARK 0.1.** By the way, we obtained the following rule for the inversion of the matrix  $S$  of the form (0.5), where  $\lambda_1, \lambda_2, \dots, \lambda_n; \mu_1, \mu_2, \dots, \mu_n$  are pairwise different complex numbers (this matrix is known as the Cauchy matrix). Starting from these numbers, we have to construct the rational function  $r$  of the form (0.1) (with some  $c: c \neq 0, \infty$ ). The entries  $h_{p,q}$  of the inverse matrix  $H = S^{-1}$  are of the form (0.17). The expression (0.17) for the inverse matrix  $S^{-1}$  may be represented in the matricial form:

$$S^{-1} = D_1 S D_2, \quad \text{where } D_1 = \text{diag} \left( \frac{1}{(r^{-1})'(\lambda_l)} \right), \quad D_2 = \text{diag} \left( \frac{1}{r'(\mu_l)} \right). \quad (0.29)$$

In particular, we established the invertibility of the Cauchy matrix  $S$ . Of course, the expression (0.17) for the entries of the inverse matrix  $S^{-1}$  differs from the expression obtained in ([BGR1]) in form only. In ([BGR1]) (Lemma 6.3 there) this expression was obtained by means of the Kramer matrix inversion rule, using explicit expression for the Cauchy determinant. (The determinant

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<sup>3</sup>  $\frac{1}{\lambda_p - \mu_q} \cdot \frac{1}{\mu_q - \lambda_t} = \frac{1}{\lambda_p - \lambda_t} \cdot \left( \frac{1}{\lambda_p - \mu_q} - \frac{1}{\lambda_t - \mu_q} \right).$

of the matrix  $S$  as well as its cofactors are Cauchy determinants). The square of the Cauchy determinant can be calculated from (0.29):

$$(\det S)^2 = \prod_{1 \leq p \leq n} (r^{-1})'(\lambda_p) \cdot \prod_{1 \leq q \leq n} r'(\mu_q). \quad (0.30)$$

REMARK 0.2. This derivation of the joint system representation can be extended to some classes of meromorphic functions.

Obtaining “system” representations of “scalar” rational functions, we made no use of anything that can not be generalized to the matricial case. Generalizations of the representations (0.9) and (0.11) to rational matrix functions have been done already. (See [S3], [GKLR], [BGR1], [BGR2], [BGRa]).

The term “system representation” is related to the system theory. In this theory to each linear stationary time invariant dynamical system its transfer function is corresponded, which is matrix valued (and even operator valued) one. If the state space of the system is finite dimensional, then this transfer function is rational. And if the system is a SISO system (Single Input, Single Output), then its transfer function is scalar (complex valued). This transfer function is expressed in terms of the input, output and state space operators of the system. Inversely, each rational matrix function may be represented as the transfer function of an appropriate linear time invariant system with finite dimensional state space. Such a representation<sup>4</sup> is said to be the system realization or the system representation of the given rational function.

The relationship between rational functions and linear systems can be exploited in both directions. We may apply results from matrix functions theory for study of linear systems. And we may use the system representation as a tool for study of rational matrix functions or as a tool to specify them.

It should be mentioned that it was M. Livšić who discovered the characteristic matrix function of linear operators and operator colligations (=operators nodes). He was the first to relate invariant subspaces of a linear operator and factors of its characteristic matrix function, [L2], [L3], [LP]. In the system theory language, the characteristic function of an operator node is just the transfer function of the appropriate linear time invariant system. This was shown by M.S. Livšić in [L8], [L9], [BrL]. He has also related the characteristic matrix function with the scattering theory ([L5], [L6], [L9], [BrL]).

The main goal of this paper is to present some basic results on the system representation of rational matrix function in a self-contained form. For the clarity of presentation, we restrict our consideration to the simple but important case of a rational matrix function in general position. The presentation is oriented to a “traditional” analyst. No previous knowledge in realization theory of matrix functions or its ideology is assumed. We leave detailed historical remarks for Section 5.

## 1. RATIONAL MATRIX FUNCTIONS IN GENERAL POSITION

DEFINITION 1.1. *Let  $M(z)$  be a  $k \times k$  matrix function holomorphic in a punctured neighborhood of some point  $a$  (i.e. the point  $a$  is an isolated singularity of the function  $M$ ). The point  $a$  is said to be a simple pole of the matrix function  $M$  if*

$$M(z) = \frac{M_a}{z - a} + H(z), \quad (1.1)$$

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<sup>4</sup>It looks like the representation (0.9)

where  $M_a$  is a constant matrix and the matrix function  $H$  is holomorphic at the point  $a$ . The matrix  $M_a$  is said to be the residue of the matrix function  $M$  at the point  $a$ .

DEFINITION 1.2. Let a point  $a$  be a simple pole of a  $k \times k$  matrix function  $M$  and let the residue  $M_a$  of the function  $M$  at the point  $a$  be a matrix of rank one. As a matrix of rank one, the  $k \times k$  matrix  $M_a$  can be factorized in the form

$$M_a = f_a \cdot g_a \quad (1.2)$$

where  $f_a$  is  $k$  vector-columns (i.e.  $k \times 1$  matrix),  $g_a$  is  $k$  vector-row (i.e.  $1 \times k$  matrix),  $f_a \neq 0, g_a \neq 0$ . The vectors  $f_a$  and  $g_a$  are said to be respectively the left semi-residual vector and the right semi-residual vector at the point  $a$ . The vectors  $f_a$  and  $g_a$  are defined uniquely, up to a constant factor: we can represent the number 1 in the form  $1 = d \cdot d^{-1}$  and then redistribute the factors  $d$  and  $d^{-1}$ :

$$f_a \rightarrow f_a \cdot d; \quad g_a \rightarrow d^{-1} \cdot g_a. \quad (1.3)$$

We emphasize that the notions of left and right semi-residual vectors are defined only for a simple pole with residue of rank one.

DEFINITION 1.3. A rational matrix function  $R$  ( $R \in \mathcal{R}(\mathfrak{M}_k)$ ) is said to be a rational matrix function in general position if:

1. The pole set  $\mathcal{P}(R)$  and the zero set  $\mathcal{N}(R)$  do not intersect;

$$\mathcal{P}(R) \cap \mathcal{N}(R) = \emptyset; \quad (1.4)$$

2. All poles of the function  $R$  are simple, and the residues at these poles are matrices of rank one;
3. All poles of the function  $R^{-1}$  are simple, and the residues at these poles are matrices of rank one;
4. Both functions  $R$  and  $R^{-1}$  are holomorphic at the point  $z = \infty$ .

Let  $R$  be a rational matrix function in general position, with the pole set  $\mathcal{P}(R)$  and the zero set  $\mathcal{N}(R)$ . For  $\lambda \in \mathcal{P}(R)$ ,  $\mu \in \mathcal{N}(R)$ , let

$$R_\lambda = f_\lambda \cdot g_\lambda, \quad (f_\lambda \neq 0, g_\lambda \neq 0); \quad R_\mu = f_\mu \cdot g_\mu \quad (f_\mu \neq 0, g_\mu \neq 0) \quad (1.5)$$

be factorizations of the residue  $R_\lambda$  of the function  $R$  at the point  $\lambda$  and the residue  $R_\mu$  of the function  $R^{-1}$  at the point  $\mu$  respectively, where  $f_\lambda, g_\lambda; f_\mu, g_\mu$  are the appropriate semi-residual vectors. The additive expansions

$$R(z) = R(\infty) + \sum_{\lambda \in \mathcal{P}(R)} \frac{R_\lambda}{z - \lambda} \quad (1.6)$$

$$R^{-1}(z) = R^{-1}(\infty) + \sum_{\mu \in \mathcal{N}(R)} \frac{R_\mu}{z - \mu} \quad (1.7)$$

can be rewritten in the form

$$R(z) = R(\infty) + \sum_{\lambda \in \mathcal{P}(R)} f_\lambda \cdot \frac{1}{z - \lambda} \cdot g_\lambda, \quad (1.8)$$

$$R^{-1}(z) = R^{-1}(\infty) + \sum_{\mu \in \mathcal{N}(R)} f_{\mu} \cdot \frac{1}{z - \mu} \cdot g_{\mu}, \quad (1.9)$$

LEMMA 1.1. I. Let a matrix function  $R$  be analytic in a punctured neighborhood of a point  $\lambda \in \mathbb{C}$ ,  $\lambda$  be a simple pole of the function  $R$  and a holomorphicity point of the function  $R^{-1}$ , and let  $R_{\lambda}$  be the residue of the function  $R$  at the point  $\lambda$ . Then the point  $\lambda$  is a simple pole for the “logarithmic derivative”  $R' \cdot R^{-1}$ , and for the residue  $P_{\lambda}$  of this “logarithmic derivative” at this point the conditions

$$P_{\lambda}^2 = -P_{\lambda}, \quad \text{rank} P_{\lambda} = \text{rank} R_{\lambda} \quad (1.10)$$

hold; in particular,

$$\text{trace} P_{\lambda} = -\text{rank} R_{\lambda}. \quad (1.11)$$

II. Let a matrix function  $R^{-1}$  be analytic in a punctured neighborhood of a point  $\mu \in \mathbb{C}$ ,  $\mu$  be a simple pole of the function  $R^{-1}$  and a holomorphicity point of the function  $R$ , and let  $R_{\mu}$  be the residue of the function  $R^{-1}$  at the point  $\mu$ . Then the point  $\mu$  is a simple pole for the “logarithmic derivative”  $R' \cdot R^{-1}$ , and for the residue  $P_{\mu}$  of this “logarithmic derivative” at this point the conditions

$$P_{\mu}^2 = P_{\mu}, \quad \text{rank} P_{\mu} = \text{rank} R_{\mu} \quad (1.12)$$

hold; in particular,

$$\text{trace} P_{\mu} = \text{rank} R_{\mu}. \quad (1.13)$$

PROOF. We prove only statement I of Lemma. Statement II can be proved analogously. Let

$$R(z) = \frac{R_{\lambda}}{z - \lambda} + A_0 + A_1(z - \lambda) + A_2(z - \lambda)^2 + \dots \quad (1.14)$$

and

$$R^{-1}(z) = B_0 + B_1(z - \lambda) + B_2(z - \lambda)^2 + \dots \quad (1.15)$$

be the Laurent expansions of the functions  $R$  and  $R^{-1}$  respectively. Then

$$R'(z) = -\frac{R_{\lambda}}{(z - \lambda)^2} + A_1 + 2A_2(z - \lambda) + \dots \quad (1.16)$$

Multiplying the Laurent expansions term by term, we obtain from (1.15) and (1.16)

$$R'(z) \cdot R^{-1}(z) = -\frac{R_{\lambda}B_0}{(z - \lambda)^2} - \frac{R_{\lambda}B_1}{z - \lambda} + (-RB_2 + A_1B_0) + O(z - \lambda). \quad (1.17)$$

Substituting the Laurent expansions (1.14), (1.15) into the identity  $R^{-1}(z)R(z) = I$  and multiplying these expansions term by term, we obtain

$$B_0R_{\lambda} = 0 \quad (1.18)$$

$$B_1R_{\lambda} + B_0A_0 = I. \quad (1.19)$$

Analogously, from the identity  $R(z)R^{-1}(z) = I$  we derive

$$R_{\lambda}B_0 = 0 \quad (1.20)$$

$$R_{\lambda}B_1 + A_0B_0 = I. \quad (1.21)$$



Let us examine the expansion (1.17). According to (1.20), the term  $-\frac{R_\lambda B_0}{(z-\lambda)^2}$  vanishes. Thus, the point  $\lambda$  is a simple pole for the function  $R'(z)R^{-1}(z)$ , with the residue  $P_\lambda$ ,

$$P_\lambda = -R_\lambda B_1. \quad (1.22)$$

From (1.21) and (1.22) it follows that  $I + P_\lambda = A_0 B_0$ . Hence,

$$(I + P_\lambda)P_\lambda = (A_0 B_0) \cdot (-R_\lambda B_1) = -A_0(B_0 R_\lambda)B_1.$$

According to (1.18),  $B_0 R_\lambda = 0$ . Thus  $(I + P_\lambda)P_\lambda = 0$ , i.e.  $P_\lambda^2 = -P_\lambda$ . Finally,  $P_\lambda R_\lambda = -(I - A_0 B_0)R_\lambda = -R_\lambda + A_0(B_0 R_\lambda)$ . Since  $B_0 R_\lambda = 0$ ,

$$R_\lambda = -P_\lambda R_\lambda \quad (1.23)$$

From (1.22) and (1.23) it follows that  $\text{rank} P_\lambda = \text{rank} R_\lambda$ .  $\square$

REMARK 1.1. From (1.19) and (1.20) it follows that  $R_\lambda B_1 R_\lambda = R_\lambda$ . Since  $B_1 = (R^{-1})'(\lambda)$  (see 1.15), it can be written as

$$R_\lambda (R^{-1})'(\lambda) R_\lambda = R_\lambda \quad (\lambda \in \mathcal{P}). \quad (1.24)$$

Analogously, we derive

$$R_\mu R'(\mu) R_\mu = R_\mu \quad (\mu \in \mathcal{N}). \quad (1.25)$$

We shall use the relations (1.24) and (1.25) in the following section.  $\square$

LEMMA 1.2. For a rational matrix function  $R$  in general position,

$$\#\mathcal{P}(R) = \#\mathcal{N}(R) \quad (\stackrel{\text{def}}{=} n(R)). \quad (1.26)$$

PROOF. To prove the statement of Lemma, we have to consider, in some way or another, the “logarithmic derivative”  $R'(z) \cdot R^{-1}(z)$ , (or  $R^{-1}(z) \cdot R'(z)$ ). For a rational matrix function  $R$  in general position, its logarithmic derivative  $R' \cdot R^{-1}$  admits the expansion

$$R'(z) \cdot R^{-1}(z) = \sum_{\lambda \in \mathcal{P}(R)} \frac{P_\lambda}{z - \lambda} + \sum_{\mu \in \mathcal{N}(R)} \frac{P_\mu}{z - \mu}. \quad (1.27)$$

Indeed, the (rational) function  $R'(z) \cdot R^{-1}(z)$  may have singularities only at the points of the set  $\mathcal{P}(R) \cup \mathcal{N}(R)$ . According to item 4 of Definition 1.3, the logarithmic derivative vanishes at the point  $\infty$ ; moreover,

$$R'(z) \cdot R^{-1}(z) = O(|z|^{-2}) \quad (z \rightarrow \infty) \quad (1.28)$$

According to Lemma 1.1, all singularities of the logarithmic derivative are simple poles with residues  $P_\lambda$  and  $P_\mu$  of rank one ( $\lambda \in \mathcal{P}(R)$ ,  $\mu \in \mathcal{N}(R)$  respectively). Thus, the expansion (1.27) holds, with  $\text{trace} P_\lambda = -1$ ,  $\text{trace} P_\mu = 1$ . From (1.27) and (1.28) it follows that

$$\sum_{\lambda \in \mathcal{P}(R)} P_\lambda + \sum_{\mu \in \mathcal{N}(R)} P_\mu = 0. \quad (1.29)$$

Because, according to Lemma 1.1,  $\text{trace} P_\lambda = -1$ , ( $\lambda \in \mathcal{P}(R)$ ),  $\text{trace} P_\mu = 1$ , ( $\mu \in \mathcal{N}(R)$ ), from (1.29) it follows that  $\sum_{\lambda \in \mathcal{P}(R)} (-1) + \sum_{\mu \in \mathcal{N}(R)} 1 = 0$ . This equality coincides with (1.26). Of course,

this result could be obtained using the operator version of Rouché theorem from [GS].  $\square$

DEFINITION 1.4. For a rational  $k \times k$  matrix function  $R$  in general position, let us order its poles and “zeros” somehow:  $\mathcal{P} = \{\lambda_1, \dots, \lambda_n\}, \mathcal{N} = \{\mu_1, \dots, \mu_n\}$ . (We remind that  $\# \mathcal{P}(R) = \# \mathcal{N}(R)$ .) We introduce the  $n \times n$  diagonal matrices

$$A_{\mathcal{P}} = \text{diag}(\lambda_1, \dots, \lambda_n), \quad A_{\mathcal{N}} = \text{diag}(\mu_1, \dots, \mu_n), \quad (1.30)$$

$k \times n$  matrices  $F_{\mathcal{P}}, F_{\mathcal{N}}$  and  $n \times k$  matrices  $G_{\mathcal{P}}, G_{\mathcal{N}}$ :

$$F_{\mathcal{P}} = [f_{\lambda_1}, \dots, f_{\lambda_n}], \quad F_{\mathcal{N}} = [f_{\mu_1}, \dots, f_{\mu_n}], \quad (1.31)$$

$$G_{\mathcal{P}} = \begin{bmatrix} g_{\lambda_1} \\ \vdots \\ g_{\lambda_n} \end{bmatrix}, \quad G_{\mathcal{N}} = \begin{bmatrix} g_{\mu_1} \\ \vdots \\ g_{\mu_n} \end{bmatrix}, \quad (1.32)$$

where  $f_{\lambda_j}, g_{\lambda_j}$  ( $\lambda_j \in \mathcal{P}$ ) are the left and right semi-residual vectors at the pole  $\lambda_j$  of  $R$ ,  $f_{\mu_j}, g_{\mu_j}$  ( $\mu_j \in \mathcal{N}$ ) are the left and right semi-residual vectors at the pole  $\mu_j$  of  $R^{-1}$ .

The matrices  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  are said to be the pole matrix and the zero matrix respectively for the matrix function  $R$ .

The matrices  $F_{\mathcal{P}}$  and  $G_{\mathcal{P}}$  are said to be the left- and the right semi-residual matrices corresponding to the pole set  $\mathcal{P}(R)$ .

The matrices  $F_{\mathcal{N}}$  and  $G_{\mathcal{N}}$  are said to be the left- and the right semi-residual matrices corresponding to the zero set  $\mathcal{N}(R)$ .  $\square$

REMARK 1.2. It should be mentioned that if we order somehow the poles and the zeros, then the pole and the zero matrices  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  are defined uniquely, and the semi-residual matrices  $F_{\mathcal{P}}, G_{\mathcal{P}}, F_{\mathcal{N}}, G_{\mathcal{N}}$  are defined essentially uniquely, up to multiplication by diagonal matrices with non-zero diagonal entries:

$$F_{\mathcal{P}} \rightarrow F_{\mathcal{P}} \cdot D_{\mathcal{P}}, \quad G_{\mathcal{P}} \rightarrow D_{\mathcal{P}}^{-1} \cdot G_{\mathcal{P}}, \quad (1.33)$$

$$F_{\mathcal{N}} \rightarrow F_{\mathcal{N}} \cdot D_{\mathcal{N}}, \quad G_{\mathcal{N}} \rightarrow D_{\mathcal{N}}^{-1} \cdot G_{\mathcal{N}}, \quad (1.34)$$

where

$$D_{\mathcal{P}} = \text{diag}(d_{1,\mathcal{P}}, \dots, d_{n,\mathcal{P}}) \quad (d_{j,\mathcal{P}} \neq 0, j = 1, 2, \dots, n), \quad (1.35)$$

$$D_{\mathcal{N}} = \text{diag}(d_{1,\mathcal{N}}, \dots, d_{n,\mathcal{N}}) \quad (d_{j,\mathcal{N}} \neq 0, j = 1, 2, \dots, n). \quad (1.36)$$

This freedom in choice of these diagonal matrices  $D_{\mathcal{P}}, D_{\mathcal{N}}$  can be used to simplify some formulas. Of course, for given left semi-residual matrix  $F_{\mathcal{P}}$ , the right semi-residual matrix  $G_{\mathcal{P}}$  is determined uniquely; for given right semi-residual matrix  $G_{\mathcal{P}}$ , the left semi-residual matrix  $F_{\mathcal{P}}$  is determined uniquely, etc.  $\square$

It is clear that the additive expansions (1.8) and (1.9) may be rewritten in the matricial form

$$R(z) = R(\infty) + F_{\mathcal{P}} \cdot (zI - A_{\mathcal{P}})^{-1} \cdot G_{\mathcal{P}}, \quad (1.37)$$

$$R^{-1}(z) = R^{-1}(\infty) + F_{\mathcal{N}} \cdot (zI - A_{\mathcal{N}})^{-1} \cdot G_{\mathcal{N}}. \quad (1.38)$$

## 2. THE JOINT REPRESENTATION OF THE KERNELS ASSOCIATED WITH A RATIONAL MATRIX FUNCTION IN GENERAL POSITION.

DEFINITION 2.1. Given a rational matrix function  $R$  of one variables, we associate with it two matrix function of two variables,  $K_R^r$  and  $K_R^l$ :

$$K_R^r(x, y) = \frac{R(x)R^{-1}(y) - I}{x - y} \quad (2.1)$$

and

$$K_R^l(x, y) = \frac{R^{-1}(x)R(y) - I}{x - y} \quad (2.2)$$

The function  $K_R^r$  is said to be the right kernel associated with the function  $R$ .

The function  $K_R^l$  is said to be the left kernel associated with the function  $R$ .  $\square$

REMARK 2.1. If  $P$  is a polynomial, the expression  $B(x, y) = \frac{P(x) - P(y)}{x - y}$  is said to be the Bezoutiant of the polynomial  $P$ . The expressions (2.1) and (2.2) look like a Bezoutiant.  $\square$

THEOREM 2.1. Let  $R$  be a rational  $k \times k$  matrix function in general position;  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  be the pole and the zero matrices for  $R$ ;  $F_{\mathcal{P}}$  and  $G_{\mathcal{N}}$  be the left pole and the right zero semiresidual matrices respectively.

Then:

1. The right kernel  $K_R^r$  is representable in the form

$$K_R^r(x, y) = F_{\mathcal{P}}(xI - A_{\mathcal{P}})^{-1} H^r (yI - A_{\mathcal{N}})^{-1} G_{\mathcal{N}}, \quad (2.3)$$

where  $H^r$  is some  $n \times n$  matrix.

2. For given matrices <sup>5</sup>  $F_{\mathcal{P}}$  and  $G_{\mathcal{N}}$ , the matrix  $H^r$  is defined uniquely:

$$H^r = \|h_{p,q}^r\|_{1 \leq p,q \leq n}, \quad h_{p,q}^r = \frac{g_{\lambda_p} \cdot f_{\mu_q}}{\lambda_p - \mu_q}. \quad (2.4)$$

3. The matrix  $H^r = \|h_{p,q}^r\|$  is invertible.

The left version of this theorem holds as well.

THEOREM 2.2. Let  $R$  be a rational  $k \times k$  matrix function in general position;  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  be the pole and the zero matrices for  $R$ ;  $F_{\mathcal{N}}$  and  $G_{\mathcal{P}}$  be the left zero and the right pole semiresidual matrices respectively.

Then:

1. The left kernel  $K_R^l$  is representable in the form

$$K_R^l(x, y) = F_{\mathcal{N}}(xI - A_{\mathcal{N}})^{-1} H^l (yI - A_{\mathcal{P}})^{-1} G_{\mathcal{P}}, \quad (2.5)$$

where  $H^l$  is some  $n \times n$  matrix.

2. For given matrices <sup>5</sup>  $F_{\mathcal{N}}$  and  $G_{\mathcal{P}}$ , the matrix  $H^l$   $H^l = \|h_{p,q}^l\|_{1 \leq p,q \leq n}$  is defined uniquely:

$$H^l = \|h_{p,q}^l\|_{1 \leq p,q \leq n}, \quad h_{p,q}^l = \frac{g_{\mu_p} \cdot f_{\lambda_q}}{\mu_p - \lambda_q}. \quad (2.6)$$

3. The matrix  $H^l$  is invertible.

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<sup>5</sup>We remind, that the matrices  $F_{\mathcal{P}}$ ,  $G_{\mathcal{P}}$ ,  $F_{\mathcal{N}}$ ,  $G_{\mathcal{N}}$  are defined only up to transformations (1.33), (1.34) with arbitrary diagonal invertible matrices  $D_{\mathcal{P}}$ ,  $D_{\mathcal{N}}$ .

DEFINITION 2.2. The matrices  $H^r$  and  $H^l$  which appear in the representations (2.3) and (2.5) of the kernels  $K_R^r$  and  $K_R^l$ , are said to be the right core matrix and the left core matrix respectively.  $\square$

COROLLARY 2.1. Let  $R$  be a rational  $k \times k$  matrix function in general position;  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  be the pole and the zero matrices for  $R$ ;  $F_{\mathcal{P}}, F_{\mathcal{N}}, G_{\mathcal{P}}, G_{\mathcal{N}}$  are the appropriate semi-residual matrices;  $H^r, H^l$  are the appropriate core matrices.

Then the matrices  $R(x) \cdot R^{-1}(y)$  and  $R^{-1}(x) \cdot R(y)$  admit the representations

$$R(x)(R(y))^{-1} = I + (x - y) F_{\mathcal{P}} (xI - A_{\mathcal{P}})^{-1} H^r (yI - A_{\mathcal{N}})^{-1} G_{\mathcal{N}}, \quad (2.7)$$

$$(R(x))^{-1} R(y) = I + (x - y) F_{\mathcal{N}} (xI - A_{\mathcal{N}})^{-1} H^l (yI - A_{\mathcal{P}})^{-1} G_{\mathcal{P}}. \quad (2.8)$$

Under the normalizing condition

$$R(\infty) = I, \quad (2.9)$$

the matrix functions  $R, R^{-1}$  themselves admit the representations

$$R(z) = I - F_{\mathcal{P}} (zI - A_{\mathcal{P}})^{-1} H^r G_{\mathcal{N}}, \quad (2.10)$$

$$R^{-1}(z) = I + F_{\mathcal{P}} H^r (zI - A_{\mathcal{N}})^{-1} G_{\mathcal{N}}, \quad (2.11)$$

$$R(z) = I + F_{\mathcal{N}} H^l (zI - A_{\mathcal{P}})^{-1} G_{\mathcal{P}}, \quad (2.12)$$

$$R^{-1}(z) = I - F_{\mathcal{N}} (zI - A_{\mathcal{N}})^{-1} H^l G_{\mathcal{P}}. \quad (2.13)$$

PROOF. The representations (2.7) and (2.8) are nothing more than the representations (2.3) and (2.5) rewritten in the terms of the functions  $R(x) \cdot R^{-1}(y)$  and  $R^{-1}(x) \cdot R(y)$ . Letting  $y$  tend to  $\infty$  in (2.7), we obtain (2.10); letting  $x$  tend to  $\infty$  in (2.7), we obtain (2.11); Letting  $y$  tend to  $\infty$  in (2.8), we obtain (2.12); letting  $x$  tend to  $\infty$  in (2.8), we obtain (2.13).  $\square$

THEOREM 2.3. Let  $R$  be a rational  $k \times k$  matrix function in general position;  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  be the pole and the zero matrices for  $R$ ;  $F_{\mathcal{P}}, F_{\mathcal{N}}, G_{\mathcal{P}}, G_{\mathcal{N}}$  are the appropriate semi-residual matrices;  $H^r, H^l$  are the appropriate core matrices.

Then these matrices are involved in the Sylvester-Lyapunov equalities:

$$A_{\mathcal{P}} H^r - H^r A_{\mathcal{N}} = G_{\mathcal{P}} F_{\mathcal{N}}, \quad (2.14)$$

$$A_{\mathcal{N}} H^l - H^l A_{\mathcal{P}} = G_{\mathcal{N}} F_{\mathcal{P}}. \quad (2.15)$$

PROOF. The matrices  $G_{\mathcal{P}} F_{\mathcal{N}}$  and  $G_{\mathcal{N}} F_{\mathcal{P}}$  are of the form

$$G_{\mathcal{P}} \cdot F_{\mathcal{N}} = \|g_{\lambda_p} \cdot f_{\mu_q}\|_{1 \leq p, q \leq n}, \quad G_{\mathcal{N}} F_{\mathcal{P}} = \|g_{\mu_p} \cdot f_{\lambda_q}\|_{1 \leq p, q \leq n}. \quad (2.16)$$

The assertion of Theorem 2.3 follows from the explicit expressions (2.16), (2.4), (2.6), (1.30).  $\square$

THEOREM 2.4. Let  $R$  be a rational  $k \times k$  matrix function in general position, which satisfies the normalizing condition (2.9);  $F_{\mathcal{P}}, F_{\mathcal{N}}, G_{\mathcal{P}}, G_{\mathcal{N}}$  are the appropriate semi-residual matrices;  $H^r, H^l$  are the appropriate core matrices.

Then these matrices are involved in the equalities

$$\text{a). } H^r G_{\mathcal{N}} = -G_{\mathcal{P}}; \quad \text{b). } H^l G_{\mathcal{P}} = -G_{\mathcal{N}}; \quad \text{c). } F_{\mathcal{P}} H^r = F_{\mathcal{N}}; \quad \text{d). } F_{\mathcal{N}} H^l = F_{\mathcal{P}} \quad (2.17)$$

We give two proofs of Theorem 2.4.

PROOF I. We compare the formulas (1.37) and (2.10). In the additive representation (1.37), for given left semi-residual matrix  $F_{\mathcal{P}}$ , the right one  $G_{\mathcal{P}}$  is determined uniquely. (See Remark 1.2). Therefore, (2.17.a) holds. Analogously, comparing the formulas (1.37) and (2.12), we obtain (2.17.d). Comparing (1.38) and (2.11), (2.13), we obtain (2.17.c) and (2.17.b).  $\square$

PROOF II. We proof only the equality (2.17.a). The equality (2.17.c) can be proved analogously. (2.17.b) follows from (2.17.a) and (2.31), etc. The matricial equality (2.17.a) is the same that the system of  $n$  scalar equalities

$$\sum_{1 \leq q \leq n} h_{p,q}^r g_{\mu_q} = -g_{\lambda_p}, \quad p = 1, 2, \dots, n. \quad (2.18)$$

Since  $f_{\lambda_p} \neq 0$ , the last equality is equivalent to the equality

$$f_{\lambda_p} \cdot \left( \sum_{1 \leq q \leq n} h_{p,q}^r \right) \cdot g_{\mu_q} = -f_{\lambda_p} \cdot g_{\lambda_p}. \quad (2.19)$$

Substituting (2.4) into (2.19), we come to equality

$$f_{\lambda_p} \cdot \left( \sum_{1 \leq q \leq n} \frac{g_{\lambda_p} f_{\mu_q}}{\lambda_p - \mu_q} \right) \cdot g_{\mu_q} = -f_{\lambda_p} \cdot g_{\lambda_p}, \quad (2.20)$$

which is the same that the equality

$$f_{\lambda_p} g_{\lambda_p} \cdot \left( \sum_{1 \leq q \leq n} \frac{f_{\mu_q} g_{\mu_q}}{\lambda_p - \mu_q} \right) = -f_{\lambda_p} g_{\lambda_p}. \quad (2.21)$$

Since  $f_{\lambda_p} g_{\lambda_p} = R_{\lambda_p}$ ,  $f_{\mu_q} g_{\mu_q} = R_{\mu_q}$ , the last equality takes the form

$$R_{\lambda_p} \cdot \left( I + \sum_{1 \leq q \leq n} \frac{R_{\mu_q}}{\lambda_p - \mu_q} \right) = 0. \quad (2.22)$$

According to (1.7),

$$I + \sum_{1 \leq q \leq n} \frac{R_{\mu_q}}{\lambda_p - \mu_q} = R^{-1}(\lambda_p).$$

Hence, (2.22) takes the form  $R_{\lambda_p} \cdot R^{-1}(\lambda_p) = 0$ . According to (1.20), this equality is true.  $\square$

The proofs of theorems 2.1 and 2.2 are analogous. We will prove only the first one of them.

PROOF of Theorem 2.1 . I. First of all, we obtain the representation (2.3). The main idea of the proof is to expand the function  $K_R^r$  of two variables into a double simple fraction series and then to interpret this expansion as the matricial equality (2.3). Actually, we derive not a *double* expansion of a function of two variables, but an *iterated* one. Let us fix a point  $y \in \overline{\mathbb{C}} \setminus \mathcal{N}(R)$ . For this fixed value  $y$ , we consider the kernel  $K_R^r(x, y)$  as a function of the variable  $x$ . This function is rational with respect to  $x$ , and  $K_R^r(x, y) \rightarrow 0$  by  $x \rightarrow \infty$ . It may have singularities only at the points of the set  $\mathcal{P}(R)$  and at the point  $y$  where denominator  $x - y$  vanishes. Actually this function is holomorphic at the point  $x = y$  because the numerator vanishes at the point  $x = y$  as well. At each of the points  $\lambda_p \in \mathcal{P}(R)$  the function  $K_R^r(x, y)$  (considered as a function of  $x$ ) either is holomorphic or has a simple pole with residue  $K_{\lambda_p}^r(y)$  of rank one:

$$K_{\lambda_p}^r(y) = \frac{R_{\lambda_p} \cdot R^{-1}(y)}{\lambda_p - y}, \quad (1 \leq p \leq n). \quad (2.23)$$

Expanding the function  $K_R^r(x, y)$  into the simple fraction sum, we obtain

$$K_R^r(x, y) = \sum_{1 \leq p \leq n} K_{\lambda_p}^r(y) \cdot \frac{1}{x - \lambda_p}. \quad (2.24)$$

In its turn the residue  $K_{\lambda_p}^r(y)$ , considered as a function of  $y$ , is a rational function. It vanishes at the point  $\infty$ . This function is also holomorphic at the point  $\lambda_p$  because the numerator  $R_{\lambda_p} \cdot R^{-1}(y)$  vanishes at the point  $\lambda_p$ : the equality  $R_{\lambda_p} \cdot R^{-1}(\lambda_p) = 0$  is the same as the equality  $R_{\lambda} B_0 = 0$  in (1.20). Thus, the only possible singularities of the function  $K_{\lambda_p}^r(y)$  are the points  $\mu_q$  ( $q = 1, \dots, n$ ) of the set  $\mathcal{N}(R)$ . These singularities are simple poles, with the residues  $K_{p,q}^r$ :

$$K_{p,q}^r = \frac{R_{\lambda_p} \cdot R_{\mu_q}}{\lambda_p - \mu_q}, \quad (1 \leq p, q \leq n). \quad (2.25)$$

Expanding the function  $K_{\lambda_p}^r$  into the simple fraction sum, we obtain

$$K_{\lambda_p}^r(y) = \sum_{1 \leq q \leq n} K_{p,q}^r \cdot \frac{1}{y - \mu_q}. \quad (2.26)$$

Combining (2.24) and (2.26) (and transforming the iterated sum into the double sum), we obtain the double expansion

$$K_R^r(x, y) = \sum_{\substack{1 \leq p \leq n \\ 1 \leq q \leq n}} \frac{1}{x - \lambda_p} \cdot K_{p,q}^r \cdot \frac{1}{y - \mu_q}. \quad (2.27)$$

Substituting into (2.25) expressions (1.5), we get

$$K_{p,q}^r = f_{\lambda_p} \cdot h_{p,q}^r \cdot g_{\mu_q}, \quad (2.28)$$

where  $h_{p,q}^r$  are defined by (2.4). Thus, the expansion (2.27) takes the form

$$K_R^r(x, y) = \sum_{\substack{1 \leq p \leq n \\ 1 \leq q \leq n}} \frac{f_{\lambda_p}}{x - \lambda_p} \cdot h_{p,q}^r \cdot \frac{g_{\mu_q}}{y - \mu_q}, \quad (2.29)$$

where  $f_{\lambda_p}$  and  $g_{\mu_q}$  are the left semi-residual vector at the pole  $\lambda_p$  and the right semi-residual vector at the “zero”  $\mu_q$  respectively.

The representation (2.3) is simply the representation (2.29) in the matricial form. The statement 1 of Theorem 2.1 is proved.

From (2.27) it follows, that

$$K_{p,q}^r = \lim_{\substack{x \rightarrow \lambda_p \\ y \rightarrow \mu_q}} (x - \lambda_p)(y - \mu_q) \cdot K_R^r(x, y). \quad (2.30)$$

Thus, the values  $K_{p,q}^r$  are determined from the kernel  $K_R^r(x, y)$  uniquely. From (2.28) it follows, that (for given  $f_{\lambda_p}$ ,  $g_{\mu_q}$ ) the values  $h_{p,q}^r$  are determined uniquely. The statement 2 of Theorem 2.1 is proved.

The statement 3 of Theorem 2.1 follows immediately from Theorem 2.5 below, where we not only prove the invertibility of matrices  $H_r$  and  $H_l$ , but also find their inverse matrices.  $\square$

**THEOREM 2.5.** *Let  $R$  be a rational matrix function in general position, and the matrices  $H^r$ ,  $H^l$  are defined from it according to (2.4), (2.6), where  $\{\lambda_1, \dots, \lambda_n\} = \mathcal{P}(R)$ ,  $\{\mu_1, \dots, \mu_n\} = \mathcal{N}(R)\}$ ,  $f_{\lambda}$ ,  $g_{\lambda}$ ,  $f_{\mu}$ ,  $g_{\mu}$  are appropriate semi-residual vectors. Then the equality holds:*

$$H^r \cdot H^l = I, \quad H^l \cdot H^r = I \quad (2.31)$$

REMARK 2.2. If we already know from somewhere that the matrix  $H^l$  is invertible, then we can easily deduce that  $(H^l)^{-1} = H^r$ . Indeed, multiplying the equation (2.15) by the matrices  $(H^l)^{-1}$  from the both sides, from the right and from the left, and taking into account that

$$(H^l)^{-1}G_{\mathcal{N}} = -G_{\mathcal{P}}, \quad F_{\mathcal{P}}(H^l)^{-1} = F_{\mathcal{N}}$$

(these are equalities (2.17.b) and (2.17.d)) , we obtain the equality

$$A_{\mathcal{P}}(H^l)^{-1} - (H^l)^{-1}A_{\mathcal{N}} = G_{\mathcal{P}}F_{\mathcal{N}}. \quad (2.32)$$

Thus, each of the matrices  $(H^l)^{-1}$  and  $H^r$  is the solution of the same Sylvester-Lyapunov equation  $A_{\mathcal{P}}X - XA_{\mathcal{N}} = G_{\mathcal{P}}F_{\mathcal{N}}$ . The condition  $\mathcal{P}(R) \cap \mathcal{N}(R) = \emptyset$  means that  $\sigma_{A_{\mathcal{P}}} \cap \sigma_{A_{\mathcal{N}}} = \emptyset$ . Under this condition, the solution  $X$  of the Sylvester-Lyapunov equation  $A_{\mathcal{P}}X - XA_{\mathcal{N}} = G_{\mathcal{P}}F_{\mathcal{N}}$  is unique. Hence,  $(H^l)^{-1} = H^r$ .  $\square$

We give two proofs of Theorem 2.5.

PROOF I. We prove only the first equality in (2.31). Multiplying (2.14) by the matrix  $H^l$  from the right and (2.15) by the matrix  $H^r$  from the left, we came to the equalities

$$A_{\mathcal{P}}H^rH^l - H^rA_{\mathcal{N}}H^l = G_{\mathcal{P}}F_{\mathcal{N}}H^l$$

and

$$H^rA_{\mathcal{N}}H^l - H^rH^lA_{\mathcal{P}} = H^rG_{\mathcal{N}}F_{\mathcal{P}}.$$

Taking into account the equalities (2.17.a) and (2.17.d), we obtain

$$A_{\mathcal{P}}H^rH^l - H^rA_{\mathcal{N}}H^l = G_{\mathcal{P}}F_{\mathcal{P}}$$

and

$$H^rA_{\mathcal{N}}H^l - H^rH^lA_{\mathcal{P}} = -G_{\mathcal{P}}F_{\mathcal{P}}.$$

Adding two last equalities, we see that the matrices  $H^rH^l$  and  $A_{\mathcal{P}}$  commute:

$$(H^rH^l)A_{\mathcal{P}} = A_{\mathcal{P}}H^rH^l.$$

Hence,

$$(H^rH^l)\varphi(A_{\mathcal{P}}) = \varphi(A_{\mathcal{P}})(H^rH^l). \quad (2.33)$$

for every function  $\varphi$  which is holomorphic on the spectrum  $\sigma_{A_{\mathcal{P}}}$  of the matrix  $A_{\mathcal{P}}$ . From (2.17.a) and (2.17.b) it follows that

$$(H^rH^l)G_{\mathcal{P}} = G_{\mathcal{P}}.$$

Multiplying this equality by  $\varphi(A_{\mathcal{P}})$  from the left and taking into account the commutational relation (2.33), we obtain that

$$(H^rH^l - I)\varphi(A_{\mathcal{P}})G_{\mathcal{P}} = 0. \quad (2.34)$$

Let us fix an index  $q \in [1, \dots, n]$  and specify the function  $\varphi : \varphi(\lambda_p) = \delta_{p,q}$ ,  $p = 1, 2, \dots, n$ . By such choice of  $\varphi$ ,  $\varphi(A_{\mathcal{P}}) = \text{diag}[\delta_{1,q}, \delta_{2,q}, \dots, \delta_{n,q}]$ . Thus,

$$\varphi(A_{\mathcal{P}})G_{\mathcal{P}} = \begin{bmatrix} \delta_{1,q} g_{\lambda_1} \\ \vdots \\ \delta_{n,q} g_{\lambda_n} \end{bmatrix}.$$

Therefore we obtain that  $m_{p,q}g_{\lambda_q} = 0$  for  $1 \leq p, q \leq n$ , where  $M \stackrel{\text{def}}{=} H^r H^l - I$ ,  $M = \|m_{p,q}\|_{1 \leq p, q \leq n}$ . Since  $m_{p,q} \in \mathbb{C}$  and  $g_{\lambda_q}$  is a non-zero vector row,  $m_{p,q} = 0$  for all  $1 \leq p, q \leq n$ , i.e.  $M = 0$ . Hence,  $H^r H^l - I = 0$ .  $\square$

PROOF II. We prove only the first equality in (2.31). This matrix equality is equivalent to the system of  $n^2$  scalar equations

$$\sum_{1 \leq q \leq n} h_{p,q}^r \cdot h_{q,p}^l = 1, \quad p = 1, 2, \dots, n. \quad (2.35)$$

and

$$\sum_{1 \leq q \leq n} h_{p,q}^r \cdot h_{q,s}^l = 0, \quad p = 1, 2, \dots, n; \quad s = 1, 2, \dots, n; \quad p \neq s. \quad (2.36)$$

According to (2.4), (2.6) equality (2.35) means that

$$\sum_{1 \leq q \leq n} \frac{(g_{\lambda_p} f_{\mu_q}) \cdot (g_{\mu_q} f_{\lambda_p})}{(\lambda_p - \mu_q)^2} = -1. \quad (2.37)$$

Because  $f_{\lambda_p} \neq 0$ ,  $g_{\lambda_p} \neq 0$ , the last equality is equivalent<sup>6</sup> to the equality

$$f_{\lambda_p} \cdot \sum_{1 \leq q \leq n} \frac{(g_{\lambda_p} f_{\mu_q}) \cdot (g_{\mu_q} f_{\lambda_p})}{(\lambda_p - \mu_q)^2} \cdot g_{\lambda_p} = -f_{\lambda_p} \cdot g_{\lambda_p}. \quad (2.38)$$

or, what is the same<sup>7</sup>, to the equality

$$f_{\lambda_p} g_{\lambda_p} \cdot \sum_{1 \leq q \leq n} \frac{f_{\mu_q} g_{\mu_q}}{(\lambda_p - \mu_q)^2} \cdot f_{\lambda_p} g_{\lambda_p} = -f_{\lambda_p} g_{\lambda_p}. \quad (2.39)$$

Taking into account the factorization (1.5), we see, that the equality (2.39) is equivalent to the equality

$$R_{\lambda_p} \cdot \sum_{1 \leq q \leq n} \frac{R_{\mu_q}}{(\lambda_p - \mu_q)^2} \cdot R_{\lambda_p} = -R_{\lambda_p}. \quad (2.40)$$

In view of (1.7),

$$\sum_{1 \leq q \leq n} \frac{R_{\mu_q}}{(\lambda_p - \mu_q)^2} = -(R^{-1})'(\lambda_p). \quad (2.41)$$

Thus, the equality (2.40) takes the form

$$R_{\lambda_p} \cdot (R^{-1})'(\lambda_p) \cdot R_{\lambda_p} = R_{\lambda_p}. \quad (2.42)$$

According to Remark 1.1, the equality (2.42) holds. (See (1.24)). Thus, the equalities (2.35) are established.

The equalities (2.36) can be established in the same way. According to (2.4) and (2.6), the equality (2.36) means that

$$\sum_{1 \leq q \leq n} \frac{(g_{\lambda_p} f_{\mu_q}) \cdot (g_{\mu_q} f_{\lambda_s})}{(\lambda_p - \mu_q) \cdot (\lambda_s - \mu_q)} = 0, \quad p \neq s. \quad (2.43)$$

---

<sup>6</sup> If  $f$  is non-zero vector-columns,  $g$  is non-zero vector-row, then the equality  $c_1 = c_2$ , where  $c_1, c_2$  are complex numbers, is equivalent to the equality  $f c_1 g = f c_2 g$ .

<sup>7</sup> Here we use the associativity of the matrix multiplication.



Because  $f_{\lambda_p} \neq 0$ ,  $g_{\lambda_s} \neq 0$ , the last equality is equivalent (see the footnote <sup>6</sup>) to the equality

$$f_{\lambda_p} \cdot \sum_{1 \leq q \leq n} \frac{(g_{\lambda_p} f_{\mu_q}) \cdot (g_{\mu_q} f_{\lambda_s})}{(\lambda_p - \mu_q) \cdot (\lambda_s - \mu_q)} \cdot g_{\lambda_s} = 0, \quad (2.44)$$

or (see the footnote <sup>7</sup>), what is the same, to the equality

$$f_{\lambda_p} g_{\lambda_p} \cdot \sum_{1 \leq q \leq n} \left( \frac{f_{\mu_q} g_{\mu_q}}{\lambda_p - \mu_q} - \frac{f_{\mu_q} g_{\mu_q}}{\lambda_s - \mu_q} \right) \cdot f_{\lambda_s} g_{\lambda_s} = 0, \quad p \neq s, \quad (2.45)$$

Taking into account the factorization (1.5), we see, that the equality (2.45) is equivalent to the equality

$$R_{\lambda_p} \cdot \sum_{1 \leq q \leq n} \left( \frac{R_{\mu_q}}{\lambda_p - \mu_q} - \frac{R_{\mu_q}}{\lambda_s - \mu_q} \right) \cdot R_{\lambda_s} = 0. \quad (2.46)$$

In view of (1.7),

$$\sum_{1 \leq q \leq n} \left( \frac{R_{\mu_q}}{\lambda_p - \mu_q} - \frac{R_{\mu_q}}{\lambda_s - \mu_q} \right) = R^{-1}(\lambda_p) - R^{-1}(\lambda_s), \quad p \neq s. \quad (2.47)$$

Thus, the equality (2.46) takes the form

$$R_{\lambda_p} \cdot \left( R^{-1}(\lambda_p) - R^{-1}(\lambda_s) \right) \cdot R_{\lambda_s} = 0. \quad (2.48)$$

In view of (1.20) and (1.18),

$$R_{\lambda_p} \cdot R^{-1}(\lambda_p) = 0, \quad R^{-1}(\lambda_s) \cdot R_{\lambda_s} = 0. \quad (2.49)$$

Thus, (2.46) holds. The equalities (2.36) are established.  $\square$

The representations (2.7) and (2.8) are almost what we need. However, there is an essential disadvantage in these representations: each one of them contains explicitly all the four semi-residual matrices. For example, the representation (2.7) contains explicitly not only the semi-residual matrices  $F_{\mathcal{P}}$  and  $G_{\mathcal{N}}$  (this is evident), but also the matrices  $F_{\mathcal{N}}$  and  $G_{\mathcal{P}}$  (see the expression (2.4) for the the right core matrix  $H^r$ ). But the four semi-residual matrices (together with zero and pole locations) are **over-determined data**: the matrix function  $R$  is completely determined by two of those semi-residual matrices only. For example, from the additive representation (1.37) it follows, that (under the normalization  $R(\infty) = I$ ) the zero and pole matrices  $A_{\mathcal{P}}$ ,  $A_{\mathcal{N}}$  together with the two semi-residual matrices  $F_{\mathcal{P}}$ ,  $G_{\mathcal{P}}$  determine completely the matrix function  $R$ , and hence the other two semi-residual matrices  $F_{\mathcal{N}}$ ,  $G_{\mathcal{N}}$ . Because of this, the semi-residual matrices  $F_{\mathcal{N}}$ ,  $G_{\mathcal{P}}$  are (at least in principle) expressible in terms of the matrices  $A_{\mathcal{P}}$ ,  $A_{\mathcal{N}}$ ,  $F_{\mathcal{P}}$ ,  $G_{\mathcal{N}}$ . Hence, we can hope to express the right core matrix  $H^r$  in terms of the matrices  $A_{\mathcal{P}}$ ,  $A_{\mathcal{N}}$ ,  $F_{\mathcal{P}}$ ,  $G_{\mathcal{N}}$ . Indeed, this can be done easily and explicitly: on the one hand, the left core matrix  $H^l$  is expressible in terms of the entries of the matrices  $A_{\mathcal{P}}$ ,  $A_{\mathcal{N}}$ ,  $F_{\mathcal{P}}$ ,  $G_{\mathcal{N}}$  only (see (2.6); on the other hand,  $H^r = (H^l)^{-1}$  (see (2.31)).

This suggests us that it may be reasonable to use the inverse matrices  $S^r$  and  $S^l$  instead of the matrices<sup>8</sup>  $H^r$  and  $H^l$ :

$$S^r \stackrel{\text{def}}{=} (H^r)^{-1} \quad \text{and} \quad S^l \stackrel{\text{def}}{=} (H^l)^{-1}. \quad (2.50)$$

---

<sup>8</sup>We remind, that, according to Theorem 2.3, the matrices  $H^r$  and  $H^l$  are mutually inverse, and hence, invertible.

The equalities (2.17) and (2.14), (2.15) can be rewritten in term of the matrices  $S^r$  and  $S^l$ . Namely, equalities (2.17) take the form

$$\text{a). } G_{\mathcal{N}} = -S^r G_{\mathcal{P}}; \quad \text{b). } G_{\mathcal{P}} = -S^l G_{\mathcal{N}}; \quad \text{c). } F_{\mathcal{P}} = F_{\mathcal{N}} S^r; \quad \text{d). } F_{\mathcal{N}} = F_{\mathcal{P}} S^l. \quad (2.51)$$

Multiplying the equality (2.14) by the matrix  $(H^r)^{-1}$  from the right and from the left and taking into account equalities (2.17.a) and (2.17.c), we transform (2.14) to the form

$$A_{\mathcal{N}} S^r - S^r A_{\mathcal{P}} = G_{\mathcal{N}} F_{\mathcal{P}}. \quad (2.52)$$

Analogously, from (2.15) and (2.17.b), (2.17.d) we derive the equality

$$A_{\mathcal{P}} S^l - S^l A_{\mathcal{N}} = G_{\mathcal{P}} F_{\mathcal{N}}. \quad (2.53)$$

Thus, the matrices  $S^r$  and  $S^l$  are solutions of the Sylvester-Lyapunov equations

$$A_{\mathcal{N}} X - X A_{\mathcal{P}} = G_{\mathcal{N}} F_{\mathcal{P}}. \quad (2.54)$$

$$A_{\mathcal{P}} X - X A_{\mathcal{N}} = G_{\mathcal{P}} F_{\mathcal{N}}. \quad (2.55)$$

respectively.

Now we change our point of view and define the matrices  $S^r$  and  $S^l$  as solutions of Sylvester-Lyapunov equations (but not as the matrices inverse to the core matrices  $H^r$  and  $H^s$ ; see (2.50)).

**DEFINITION 2.3.** Let  $R$  be a rational matrix function in general position,  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  be its pole and zero matrices,  $F_{\mathcal{P}}$ ,  $G_{\mathcal{P}}$ ,  $F_{\mathcal{N}}$ ,  $G_{\mathcal{N}}$  be its appropriate semi-residual matrices.

1. *The matrices  $S^r$  and  $S^l$  which are the solutions of the Sylvester-Lyapunov equations (2.54) and (2.55), are said to be the right zero-pole coupling matrix and the left zero-pole coupling matrix respectively.*
2. *The relations (2.51) are said to be the zero-pole coupling relations.*

**REMARK 2.3.** Since the spectra of the matrices  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  do not intersect, the Sylvester-Lyapunov equations (2.55) and (2.54) are uniquely solvable. (However, as the matrices  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  are diagonal, the solvability of these equations as well as the uniqueness is obvious). Moreover, it is possible to obtain the explicit expressions for the matrices  $S^r$  and  $S^l$  from (2.55) and (2.54):

$$\text{r). } S^r = \|s_{p,q}^r\|_{1 \leq p,q \leq n}, \quad s_{p,q}^r = \frac{g_{\mu_p} \cdot f_{\lambda_q}}{\mu_p - \lambda_q}, \quad \text{l). } S^l = \|s_{p,q}^l\|_{1 \leq p,q \leq n}, \quad s_{p,q}^l = \frac{g_{\lambda_p} \cdot f_{\mu_q}}{\lambda_p - \mu_q}. \quad (2.56)$$

(Actually, we *derived* the Sylvester-Lyapunov equations from the explicit expressions for the matrices which we interpret now as solutions of these equations).  $\square$

According to (2.50), the relations (2.31) can be rewritten in the form

$$S^r \cdot S^l = S^l \cdot S^r = I. \quad (2.57)$$

We may also refer directly to the equalities (2.37) and (2.43): these equalities mean that the matrices (2.56.r) and (2.56.l) are mutually inverse.

REMARK 2.4. The representations (2.3) and (2.5) may be rewritten in terms of the matrices  $S^r$  and  $S^l$  (instead of the matrices  $H^r$  and  $H^l$ ):

$$(R(x)(R(y))^{-1} = I + (x - y) F_{\mathcal{P}} (xI - A_{\mathcal{P}})^{-1} \cdot (S^r)^{-1} \cdot (yI - A_{\mathcal{N}})^{-1} G_{\mathcal{N}}, \quad (2.58)$$

$$(R(x))^{-1} R(y) = I + (x - y) F_{\mathcal{N}} (xI - A_{\mathcal{N}})^{-1} \cdot (S^l)^{-1} \cdot (yI - A_{\mathcal{P}})^{-1} G_{\mathcal{P}}. \quad (2.59)$$

Under the normalizing condition (2.9), the matrix functions  $R, R^{-1}$  themselves admits the representations

$$R(z) = I - F_{\mathcal{P}} (zI - A_{\mathcal{P}})^{-1} (S^r)^{-1} G_{\mathcal{N}}, \quad (2.60)$$

$$R^{-1}(z) = I + F_{\mathcal{P}} (S^r)^{-1} (zI - A_{\mathcal{N}})^{-1} G_{\mathcal{N}}, \quad (2.61)$$

$$R(z) = I + F_{\mathcal{N}} (S^l)^{-1} (zI - A_{\mathcal{P}})^{-1} G_{\mathcal{P}}, \quad (2.62)$$

$$R^{-1}(z) = I - F_{\mathcal{N}} (zI - A_{\mathcal{N}})^{-1} (S^l)^{-1} G_{\mathcal{P}}. \quad (2.63)$$

These formulas may be obtained of from (2.58), (2.59), letting  $x$  or  $y$  tend to  $\infty$  there, or from (2.10) - (2.13), rewriting them in terms of the matrices  $S^r, S^l$  (instead of the matrices  $H^r, H^l$ ).  $\square$

REMARK 2.5. Of course, we may obtain the zero-pole coupling relations (2.51) comparing the representations (1.37), (1.38) and (2.10) - (2.13).  $\square$

REMARK 2.6. The semi-residual vectors are defined not completely uniquely, but up to transformations (1.33) and (1.34) only. If the semi-residual vectors are transformed according to (1.33) and (1.34), the right hand sides of the Sylvester-Lyapunov equations (2.54) and (2.55) are transformed as:

$$G_{\mathcal{N}} F_{\mathcal{P}} \rightarrow (D_{\mathcal{N}})^{-1} \cdot G_{\mathcal{N}} F_{\mathcal{P}} \cdot D_{\mathcal{P}}; \quad G_{\mathcal{P}} F_{\mathcal{N}} \rightarrow (D_{\mathcal{P}})^{-1} \cdot G_{\mathcal{P}} F_{\mathcal{N}} \cdot D_{\mathcal{N}} \quad (2.64)$$

The solutions  $S^r$  and  $S^l$  of the Sylvester-Lyapunov equations (2.54) and (2.55) are transformed as:

$$S^r \rightarrow (D_{\mathcal{N}})^{-1} \cdot S^r \cdot D_{\mathcal{P}}; \quad S^l \rightarrow (D_{\mathcal{P}})^{-1} \cdot S^l \cdot D_{\mathcal{N}}. \quad (2.65)$$

Of course, the expressions (2.58), (2.59) (for  $R(x)(R(y))^{-1}$  and  $(R(x))^{-1}R(y)$ ) are invariant with respect to the transformations (1.33), (1.34), (2.65) (of the semi-residual and the zero-pole coupling matrices).  $\square$

The representations (2.58) and (2.59) are exactly what we need. Obtaining them is one of the main goal of this paper. Therefore we choose a special name for this representation:

DEFINITION 2.4. Let  $R$  be a rational matrix function in general position,  $A_{\mathcal{P}}, A_{\mathcal{N}}$  be its pole and zero matrices,  $F_{\mathcal{P}}, G_{\mathcal{P}}, F_{\mathcal{N}}, G_{\mathcal{N}}$  be appropriate semi-residual matrices,  $S^r, S^l$  be the solutions of the Sylvester-Lyapunov equations (2.54) and (2.55) respectively. The formulas (2.58) and (2.59) are said to be the right joint system representation of the pair  $R, R^{-1}$  and the left joint system representation of the pair  $R, R^{-1}$  respectively.

The formulas (2.60), (2.61), (2.62), (2.63) (which can be obtained from (2.58), (2.59) by passage to the limit) are said to be the right system representation of the function  $R$ , the right system representation of the function  $R^{-1}$ , the left system representation of the function  $R$ , the left system representation of the function  $R^{-1}$  respectively.

REMARK 2.7. The terminology is motivated by the so-called *system theory* or, in more detail, by the *theory of linear time invariant dynamical system*. In this theory, all the objects such as the zero and pole matrices, the semi-residual matrices, the zero-pole coupling matrices are

interpreted from the point of view of dynamical systems. This interpretation does not play any role in our considerations. We need the joint system representations as a tool to introduce a convenient coordinates in the set of all rational matrix functions (in general position).  $\square$

REMARK 2.8. In the realization theory one obtains formulas like (2.60) - (2.63) for matrix functions  $R$  and  $R^{-1}$  considered individually. In the representations (2.58), (2.59) the matrix function  $R$ ,  $R^{-1}$  are considered jointly. This is the reason for using the terminology joint system representation.  $\square$

Now we summarize the results of this section and formulate

THEOREM 2.6. *Let  $R$  be a rational matrix function in general position,  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  be its pole and zero matrices,  $F_{\mathcal{P}}$ ,  $G_{\mathcal{P}}$  be its left and right pole semi-residual matrices,  $F_{\mathcal{N}}$ ,  $G_{\mathcal{N}}$  be its left and right zero semi-residual matrices. Then:*

1. *The matrices  $S^r$ , which is a solution of the Sylvester-Lyapunov equation  $A_{\mathcal{N}}X - XA_{\mathcal{P}} = G_{\mathcal{N}}F_{\mathcal{P}}$ , and  $S^l$ , which is a solution of the Sylvester-Lyapunov equation  $A_{\mathcal{P}}X - XA_{\mathcal{N}} = G_{\mathcal{P}}F_{\mathcal{N}}$ :*

$$r). \quad S^r = \|s_{p,q}^r\|_{1 \leq p,q \leq n}, \quad s_{p,q}^r = \frac{g_{\mu_p} \cdot f_{\lambda_q}}{\mu_p - \lambda_q}, \quad l). \quad S^l = \|s_{p,q}^l\|_{1 \leq p,q \leq n}, \quad s_{p,q}^l = \frac{g_{\lambda_p} \cdot f_{\mu_q}}{\lambda_p - \mu_q}, \quad (2.56)$$

*are mutually inverse, i.e. the equalities*

$$S^r \cdot S^l = I, \quad S^l \cdot S^r = I \quad (2.57)$$

*hold. In particular, the matrices  $S^r$  and  $S^l$  are invertible.*

2. *The matrix function  $R(x)(R(y))^{-1}$  can be recovered from the data  $A_{\mathcal{P}}$ ,  $A_{\mathcal{N}}$ ,  $F_{\mathcal{P}}$ ,  $G_{\mathcal{N}}$  by the formula (2.58)*

$$(R(x)(R(y))^{-1} = I + (x - y) F_{\mathcal{P}} (xI - A_{\mathcal{P}})^{-1} \cdot (S^r)^{-1} \cdot (yI - A_{\mathcal{N}})^{-1} G_{\mathcal{N}}, \quad (2.58)$$

*i.e. the right joint system representation holds.*

3. *The matrix function  $(R(x))^{-1}R(y)$  can be recovered from the data  $A_{\mathcal{P}}$ ,  $A_{\mathcal{N}}$ ,  $F_{\mathcal{N}}$ ,  $G_{\mathcal{P}}$  by the formula (2.59),*

$$(R(x))^{-1}R(y) = I + (x - y) F_{\mathcal{N}} (xI - A_{\mathcal{N}})^{-1} \cdot (S^l)^{-1} \cdot (yI - A_{\mathcal{P}})^{-1} G_{\mathcal{P}}. \quad (2.59)$$

*i.e. the left joint system representation holds.*

4. *Under the normalizing condition (2.9), the matrix functions  $R$ ,  $R^{-1}$  themselves admit the representations*

$$R(z) = I - F_{\mathcal{P}} (zI - A_{\mathcal{P}})^{-1} (S^r)^{-1} G_{\mathcal{N}}, \quad (2.60)$$

$$R^{-1}(z) = I + F_{\mathcal{P}} (S^r)^{-1} (zI - A_{\mathcal{N}})^{-1} G_{\mathcal{N}}, \quad (2.61)$$

$$R(z) = I + F_{\mathcal{N}} (S^l)^{-1} (zI - A_{\mathcal{P}})^{-1} G_{\mathcal{P}}; \quad (2.62)$$

$$R^{-1}(z) = I - F_{\mathcal{N}} (zI - A_{\mathcal{N}})^{-1} (S^l)^{-1} G_{\mathcal{P}}. \quad (2.63)$$

5. *The zero-pole coupling relations hold:*

$$a). \quad G_{\mathcal{N}} = -S^r G_{\mathcal{P}}; \quad b). \quad G_{\mathcal{P}} = -S^l G_{\mathcal{N}}; \quad c). \quad F_{\mathcal{P}} = F_{\mathcal{N}} S^r; \quad d). \quad F_{\mathcal{N}} = F_{\mathcal{P}} S^l. \quad (2.51)$$

6. *If the representations (2.7) and (2.8) hold with some matrices  $H^r$  and  $H^s$ , then  $H^r = (S^r)^{-1}$ ,  $H^l = (S^l)^{-1}$  of necessity.*

REMARK 2.9. The representation (2.58) allows us to recover the matrix function  $R(x)(R(y))^{-1}$  from the left pole- and the right zero- semi-residual matrices  $F_{\mathcal{P}}$  and  $G_{\mathcal{N}}$ . whereas the representation (2.59) allows us to recover the matrix function  $(R(x))^{-1}R(y)$  from the right pole- and the left zero- semi-residual matrices  $F_{\mathcal{N}}$  and  $G_{\mathcal{P}}$ . However, sometimes one have needs for some “hybrid” formulas which allow to recover the matrix function  $R(x)(R(y))^{-1}$  from the right pole- and the left zero- semi-residual matrices  $F_{\mathcal{N}}$  and  $G_{\mathcal{P}}$  and the matrix function  $(R(x))^{-1}R(y)$  from the left pole- and the right zero- semi-residual matrices  $F_{\mathcal{P}}$  and  $G_{\mathcal{N}}$ . Such formulas can be easily derived from the joint system realization formulas (2.58), (2.59) combined with the zero-pole coupling relations (2.51). These “hybrid” formulas are of the form:

$$R(x) \cdot (R(y))^{-1} = I - (x - y) F_{\mathcal{N}} (S^l)^{-1} (xI - A_{\mathcal{P}})^{-1} S^l (yI - A_{\mathcal{N}})^{-1} (S^l)^{-1} G_{\mathcal{P}}, \quad (2.66)$$

$$(R(x))^{-1} \cdot R(y) = I - (x - y) F_{\mathcal{P}} (S^r)^{-1} (xI - A_{\mathcal{N}})^{-1} S^r (yI - A_{\mathcal{P}})^{-1} (S^r)^{-1} G_{\mathcal{N}}. \quad (2.67)$$

The matrix  $S^l$  can be calculated from the data:  $F_{\mathcal{N}}, G_{\mathcal{P}}, A_{\mathcal{P}}, A_{\mathcal{N}}$ ; the matrix  $S^r$  can be calculated from the data:  $F_{\mathcal{P}}, G_{\mathcal{N}}, A_{\mathcal{P}}, A_{\mathcal{N}}$ .  $\square$

### 3. FROM THE CHAIN IDENTITY TO THE SYLVESTER - LYAPUNOV EQUATION AND BACK.

The consideration of this item are concentrated around of the so-called *chain identity*. Let us give a number of definitions.

Let  $T(\cdot, \cdot)$  be a  $k \times k$  matrix function of two complex variables, with domain of definition  $\mathcal{D}_T$ ,  $\mathcal{D}_T \in \overline{\mathbb{C}} \times \overline{\mathbb{C}}$ ,  $T : \mathcal{D}_T \rightarrow \mathfrak{M}_k$ .

DEFINITION 3.1. A function  $T(\cdot, \cdot)$  of two variables is said to satisfy the chain identity if

$$T(x, y) \cdot T(y, z) = T(x, z) \quad (3.1)$$

for every  $x, y, z$  for which  $(x, y) \in \mathcal{D}_T$  and  $(y, z) \in \mathcal{D}_T$ . (In particular,  $(x, z)$  must belong to  $\mathcal{D}_T$ , if  $(x, y) \in \mathcal{D}_T$  and  $(y, z) \in \mathcal{D}_T$ ).

DEFINITION 3.2. A function  $T(\cdot, \cdot)$  of two variables is said to satisfy the diagonal unity identity if

$$T(x, x) = I \quad (3.2)$$

for every point  $x$  for which  $(x, x)$  belongs to the domain of definition  $\mathcal{D}_T$  of the function  $T$ .

A class of function  $T$  satisfying both the chain identity and the diagonal unity identity can be constructed in the following way.

DEFINITION 3.3. Let  $\Phi$  and  $\Phi^{-1}$  be a  $k \times k$  matrix functions of one variables with domains of definition  $\mathcal{D}_{\Phi}$  and  $\mathcal{D}_{\Phi^{-1}}$  respectively,  $\mathcal{D}_{\Phi} \in \overline{\mathbb{C}}$ ,  $\mathcal{D}_{\Phi^{-1}} \in \overline{\mathbb{C}}$ . Let us define the matrix function  $T_{\Phi}$  of two variables by the equality

$$T_{\Phi}(x, y) \stackrel{\text{def}}{=} \Phi(x) \cdot \Phi^{-1}(y) \quad (3.3)$$

with domain of definition

$$\mathcal{D}_T \stackrel{\text{def}}{=} \mathcal{D}_{\Phi} \times \mathcal{D}_{\Phi^{-1}}, \quad \text{i.e.} \quad \left( (x, y) \in \mathcal{D}_T \right) \Leftrightarrow \left( (x \in \mathcal{D}_{\Phi}) \& (y \in \mathcal{D}_{\Phi^{-1}}) \right). \quad (3.4)$$

The function  $T_{\Phi}$  is said to be the chain function generated by the function  $\Phi$ .

LEMMA 3.1. Let  $\Phi$  and  $\Phi^{-1}$  be a  $k \times k$  matrix functions of one variables with domains of definition  $\mathcal{D}_{\Phi}$  and  $\mathcal{D}_{\Phi^{-1}}$  respectively,  $\mathcal{D}_{\Phi} \in \overline{\mathbb{C}}$ ,  $\mathcal{D}_{\Phi^{-1}} \in \overline{\mathbb{C}}$ . Let  $T_{\Phi}$  be the chain function generated

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<sup>9</sup>We recall that  $\overline{\mathbb{C}} \stackrel{\text{def}}{=} \mathbb{C} \cup \infty$  is the extended complex plane.

by the function  $\Phi$ . If the matrix functions  $\Phi$  and  $\Phi^{-1}$  are mutually inverse, i.e. the identities  $\Phi(x) \cdot \Phi^{-1}(x) = \Phi^{-1}(x) \cdot \Phi(x) = I$  hold for all  $x \in \mathcal{D}_\Phi \cap \mathcal{D}_{\Phi^{-1}}$ , then for the matrix function  $T$  both chain identity and diagonal unity identity hold.

REMARK 3.1. Of course, Lemma 3.1 is reach in content only under condition  $\mathcal{D}_\Phi \cap \mathcal{D}_{\Phi^{-1}} \neq \emptyset$ . If this condition fails then the values  $T(x, x)$  and  $T(x, y) \cdot T(y, z)$  are defined for the empty set of arguments.

PROOF of Lemma 3.1. The diagonal unity identity expressed that the functions  $\Phi$  and  $\Phi^{-1}$  are mutually inverse. The chain identity for the function  $T_\Phi$  is the consequence of two facts: 1). The function  $\Phi$  and  $\Phi^{-1}$  are mutually inverse; 2). The matricial multiplication is associative.  $\square$

It turns out that each function  $T$  of two variables satisfying both the chain identity and the diagonal unity identity is of the form  $T_\Phi$  for some function  $\Phi$  of one variable.

THEOREM 3.1. Let  $T$  be a  $k \times k$  matrix function of two variables, which domain of definition  $\mathcal{D}_T \in \overline{\mathbb{C}} \times \overline{\mathbb{C}}$  is of the form  $\mathcal{D}_T = \mathcal{D}_1 \times \mathcal{D}_2$ , where  $\mathcal{D}_1 \in \overline{\mathbb{C}}$  and  $\mathcal{D}_2 \in \overline{\mathbb{C}}$ , with  $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \emptyset$ . If for the function  $T$  both chain identity and diagonal unity identity are satisfied, then the function  $T$  is of the form  $T = T_\Phi$ , (i.e.  $T(x, y) = \Phi(x) \cdot \Phi^{-1}(y)$ ), where  $\Phi$  and  $\Phi^{-1}$  are mutually inverse  $k \times k$  matrix functions of one variable, with  $\mathcal{D}_\Phi = \mathcal{D}_1$  and  $\mathcal{D}_{\Phi^{-1}} = \mathcal{D}_2$ .

PROOF. Let us fix an arbitrary point  $a$  belonging to the set  $\mathcal{D}_1 \cap \mathcal{D}_2$ . (We will call this point  $a$  the distinguished point.) Let us define now

$$\mathcal{D}_\Phi \stackrel{\text{def}}{=} \mathcal{D}_1, \quad \Phi(x) \stackrel{\text{def}}{=} T(x, a); \quad \mathcal{D}_{\Phi^{-1}} \stackrel{\text{def}}{=} \mathcal{D}_2, \quad \Phi^{-1}(y) \stackrel{\text{def}}{=} T(a, y). \quad (3.5)$$

The functions  $\Phi$  and  $\Phi^{-1}$  are mutually inverse: this follows from the chain and diagonal unity identities. The equality  $T(x, y) = \Phi(x) \cdot \Phi^{-1}(y)$  is the chain identity written down for the triple of the points  $x, a, y$ . In addition, we note that this function  $\Phi$  satisfies the normalizing condition  $\Phi(a) = I$ .  $\square$

DEFINITION 3.4. Let  $R$  be a rational  $k \times k$  matrix function of one variables,  $\det R \neq 0$ , and  $R^{-1}$  is the inverse (in the commonly accepted sense) matrix function; the domain of definition  $\mathcal{D}_R$  is the set of holomorphicity of the function  $R$ ; the domain of definition  $\mathcal{D}_R^{-1}$  is the set of holomorphicity of the function  $R^{-1}$ . (In other words, <sup>10</sup>  $\mathcal{D}_R = \overline{\mathbb{C}} \setminus \mathcal{P}(R)$ ,  $\mathcal{D}_R^{-1} = \overline{\mathbb{C}} \setminus \mathcal{N}(R)$ ). We associate with the function  $R$  two functions of two variables,  $T_R^r(x, y)$  and  $T_R^l(x, y)$ :

$$\text{a). } T_R^r(x, y) \stackrel{\text{def}}{=} R(x) \cdot R^{-1}(y); \quad \text{b). } T_R^l(x, y) \stackrel{\text{def}}{=} R^{-1}(x) \cdot R(y). \quad (3.6)$$

The function  $T_R^r$  is said to be the right chain function generated by  $R$ . The function  $T_R^l$  is said to be the left chain function generated by  $R$ .

REMARK 3.2. It is clear that that the function  $T_R^r$  is the function of the form  $T_\Phi$  (in the sense of the Definition 3.3) for  $\Phi = R$ , and the function  $T_R^l$  is the function  $T_\Phi$  for  $\Phi = R^{-1}$ . Thus, the right chain function generated by  $R$  is the left chain function generated by  $R^{-1}$ :

$$T_R^r(x, y) = T_{R^{-1}}^l(x, y). \quad (3.7)$$

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<sup>10</sup>We recall that  $\mathcal{P}(R)$  is the pole set of the function  $R$ ,  $\mathcal{N}(R)$  is the zero set of the function  $R$ , i.e. the pole set of the function  $R^{-1}$ .

From Lemma 3.1 and from Remark 3.2 it follows

**COROLLARY 3.1.** *Let  $R$  be a matrix function,  $\det R \neq 0$ . Then both matrix functions  $T_R^r(x, y)$  and  $T_R^l(x, y)$  (see (3.6)) satisfy the chain identity and the diagonal unity identity:*

$$T_R^r(x, y) \cdot T_R^r(y, z) \equiv T_R^r(x, z); \quad T_R^r(x, x) \equiv I. \quad (3.8)$$

$$T_R^l(z, y) \cdot T_R^l(y, x) \equiv T_R^l(z, x); \quad T_R^l(x, x) \equiv I. \quad (3.9)$$

□

From (2.1) and (2.2) it is clear that the right and left chain functions  $T_R^r$  and  $T_{R^{-1}}^l$  can be expressed from the right and left kernels:

$$T_R^r(x, y) = I + (x - y) \cdot K_R^r(x, y); \quad (3.10)$$

$$T_R^l(x, y) = I + (x - y) \cdot K_R^l(x, y). \quad (3.11)$$

Substituting in the chain identity (3.8) the expression (3.10) for  $T_R^r(x, y)$ , we obtain an identity for the right kernel  $K_R^r$ :

$$\left( I + (x - y)K_R^r(x, y) \right) \cdot \left( I + (y - z)K_R^r(y, z) \right) \equiv I + (x - z)K_R^r(x, z). \quad (3.12)$$

Removing the parentheses, we obtain the identity

$$(x - y)K_R^r(x, y) + (y - z)K_R^r(y, z) - (x - z)K_R^r(x, z) \equiv -(x - y)(y - z)K_R^r(x, y)K_R^r(y, z).$$

Dividing on  $(x - y)(y - z)$ , we come to the identity

$$\frac{K_R^r(x, y) - K_R^r(x, z)}{y - z} - \frac{K_R^r(x, z) - K_R^r(y, z)}{x - y} \equiv -K(x, y)K(y, z). \quad (3.13)$$

Assume now that the kernel  $K_R^r$  admits the representation (2.3), with some  $k \times n$  matrix  $F_{\mathcal{P}}$ ,  $n \times k$  matrix  $G_{\mathcal{N}}$  and  $n \times n$  matrices  $A_{\mathcal{P}}, A_{\mathcal{N}}, H^r$ . Substituting the expressions (2.3) into (3.11), we come to the identity

$$\begin{aligned} & F_{\mathcal{P}} \cdot (xI - A_{\mathcal{P}})^{-1} \cdot H^r \cdot \frac{(yI - A_{\mathcal{N}})^{-1} - (zI - A_{\mathcal{N}})^{-1}}{y - z} \cdot G_{\mathcal{N}} - \\ & F_{\mathcal{P}} \cdot \frac{(xI - A_{\mathcal{P}})^{-1} - (yI - A_{\mathcal{P}})^{-1}}{x - y} \cdot H^r \cdot (zI - A_{\mathcal{N}})^{-1} \cdot G_{\mathcal{N}} \equiv \\ & - F_{\mathcal{P}} \cdot (xI - A_{\mathcal{P}})^{-1} \cdot H^r \cdot (yI - A_{\mathcal{N}})^{-1} \cdot G_{\mathcal{N}} F_{\mathcal{P}} \cdot (yI - A_{\mathcal{P}})^{-1} \cdot H^r \cdot (zI - A_{\mathcal{N}})^{-1} \cdot G_{\mathcal{N}}. \end{aligned} \quad (3.14)$$

Using the Hilbert identity for resolvents, we come to the identity

$$\begin{aligned} & F_{\mathcal{P}} \cdot (xI - A_{\mathcal{P}})^{-1} \cdot H^r \cdot (yI - A_{\mathcal{N}})^{-1} \cdot (zI - A_{\mathcal{N}})^{-1} \cdot G_{\mathcal{N}} - \\ & F_{\mathcal{P}} \cdot (xI - A_{\mathcal{P}})^{-1} \cdot (yI - A_{\mathcal{P}})^{-1} \cdot H^r \cdot (zI - A_{\mathcal{N}})^{-1} \cdot G_{\mathcal{N}} \equiv \\ & F_{\mathcal{P}} \cdot (xI - A_{\mathcal{P}})^{-1} \cdot H^r \cdot (yI - A_{\mathcal{N}})^{-1} \cdot G_{\mathcal{N}} F_{\mathcal{P}} \cdot (yI - A_{\mathcal{P}})^{-1} \cdot H^r \cdot (zI - A_{\mathcal{N}})^{-1} \cdot G_{\mathcal{N}}. \end{aligned} \quad (3.15)$$

Putting the common factors outside the parentheses, we obtain

$$F_{\mathcal{P}} \cdot (xI - A_{\mathcal{P}})^{-1} \cdot M \cdot (zI - A_{\mathcal{N}})^{-1} \cdot G_{\mathcal{N}} \equiv 0, \quad (3.16)$$

where

$$M = H^r \cdot (yI - A_N)^{-1} - (yI - A_P)^{-1} \cdot H^r - H^r \cdot (yI - A_N)^{-1} \cdot G_N F_P \cdot (yI - A_P)^{-1} \cdot H^r \quad (3.17)$$

Assume moreover that  $R$  is a rational matrix function in general position. Let  $A_P$  and  $A_N$  be its pole and zero matrices (i.e. these matrices are of the form (1.30), where all the numbers  $\lambda_1, \dots, \lambda_n; \mu_1, \dots, \mu_n$  are pairwise different), and  $F_P$  and  $G_N$  be the pole left semi-residual and the zero right semi-residual matrices (in particular, they have the form (1.31), (1.32) where  $f_{\lambda_p}$  and  $g_{\mu_q}$  are non-zero  $k$  vector-columns).

From (3.16) it follows that

$$F_P \cdot \varphi(A_P) \cdot M \cdot \psi(A_N) \cdot G_N \equiv 0, \quad (3.18)$$

where  $\varphi$  and  $\psi$  are arbitrary functions which are analytic on the spectra of the matrices  $A_P$  and  $A_N$  respectively.

Let us fix two indices,  $p \in [1, \dots, n]$  and  $q \in [1, \dots, n]$  and specify two functions  $\varphi$  and  $\psi$ :

$$\varphi(\lambda_j) = \delta_{j,p} \quad \psi(\mu_j) = \delta_{j,q}, \quad (j = 1, 2, \dots, n).$$

Then

$$\varphi(A_P) = \text{diag}(\delta_{1,p}, \dots, \delta_{n,p}), \quad \psi(A_N) = \text{diag}(\delta_{1,q}, \dots, \delta_{n,q}). \quad (3.19)$$

For such functions  $\varphi$  and  $\psi$ , (3.18) becomes the form:

$$f_{\lambda_p} \cdot m_{p,q} \cdot g_{\mu_q} = 0, \quad (3.20)$$

where  $m_{p,q}$  is  $p, q$  entry of the matrix  $M$ . Since  $f_{\lambda_p}$  and  $g_{\mu_q}$  are non-zero vector-column and vector-row,  $m_{p,q} = 0$ . Thus,  $M = 0$ , or,

$$H^r \cdot (yI - A_N)^{-1} - (yI - A_P)^{-1} \cdot H^r - H^r \cdot (yI - A_N)^{-1} \cdot G_N F_P \cdot (yI - A_P)^{-1} \cdot H^r \equiv 0. \quad (3.21)$$

The “left” version of the equation (3.21) has the form

$$H^l \cdot (yI - A_P)^{-1} - (yI - A_N)^{-1} \cdot H^l - H^l \cdot (yI - A_P)^{-1} \cdot G_P F_N \cdot (yI - A_N)^{-1} \cdot H^l \equiv 0. \quad (3.22)$$

To obtain (3.22), we have to use the chain identity (3.9), the expression (3.11) for the chain function  $T_R^l$  in terms of its associated kernel  $K_R^l$ , and the representation (2.5) of this kernel. However, we may just replace in (3.21):

$$H^r \rightarrow H^l; \quad F_P \rightarrow F_N; \quad G_N \rightarrow G_P; \quad A_N \rightarrow A_P; \quad A_P \rightarrow A_N.$$

Let us examine the Laurent expansion (with respect to  $y$ ) of the function in the left hand side of (3.21):

$$\begin{aligned} H^r \cdot (yI - A_N)^{-1} - (yI - A_P)^{-1} \cdot H^r - H^r \cdot (yI - A_N)^{-1} \cdot G_N F_P \cdot (yI - A_P)^{-1} \cdot H^r = \\ (H^r A_N - A_P H^r - H^r G_N \cdot F_P H^r) y^{-2} + O(y^{-3}) \quad (y \rightarrow \infty) \end{aligned}$$

In view of (3.21), the Sylvester-Lyapunov equality holds:

$$H^r A_N - A_P H^r = H^r G_N \cdot F_P H^r. \quad (3.23)$$

Analogously, from (3.22) we can derive the equality

$$H^l A_P - A_N H^l = H^l G_P \cdot F_N H^l. \quad (3.24)$$



According to Theorem 2.1 (actually, according to Theorem 2.5), the *core matrix*  $H^r$  (see Definition 2.2) is invertible. (The first proof of the Theorem 2.5 is based on the Sylvester-Lyapunov equalities (2.14), (2.15), but the second one is independent from them). Multiplying (3.21) by the matrix  $(H^r)^{-1}$  from the right and by the matrix  $(H^r)^{-1}$  from the left and denoting, as before (see (2.50)),  $S^r \stackrel{\text{def}}{=} (H^r)^{-1}$ , we come to the equality (2.52). From (3.22) we can derive the equality (2.53) in the same way as we already derived the equality (2.52) from (3.21).

Thus, we obtained the equalities (2.52) and (2.53) in two different ways. The first one is based on the explicit expressions (2.4) and (2.6) for the core matrices  $H^r$  and  $H^s$ . This method uses essentially the specific character of a rational matrix function *in general position*. The second method works for much more broad classes of rational matrix functions. Up to certain point, the method works for *arbitrary* rational matrix function. First of all, this method uses the chain identities (3.8) and (3.9). These identities are evidently true for arbitrary matrix functions  $R$  which are non-degenerate (i.e.  $\det R \neq 0$ ). Then we use the representations (2.3), (2.5) for the kernels, associated with  $R$ . However, we nowhere use that the pole and zero matrices  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  are diagonal, with disjoint simple spectra. Actually, we have obtained the equality (3.18) for *any* rational matrix function  $R$  which associated kernel  $K_R^r$  admits the representation (2.1) with *arbitrary*  $A_{\mathcal{P}}, A_{\mathcal{N}}, F_{\mathcal{P}}, G_{\mathcal{N}}$  and  $H^r$  (or, what is the same, for *any* rational matrix function  $R$ , such that the function  $R(x) \cdot (R(y))^{-1}$  admits the representation (2.7) with *arbitrary*  $A_{\mathcal{P}}, A_{\mathcal{N}}, F_{\mathcal{P}}, G_{\mathcal{N}}$  and  $H^r$ ). Then we have to conclude from (3.18) (under the assumption that this equality holds for arbitrary functions  $\varphi, \psi$  which are holomorphic on the spectra of  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  respectively), that (3.21) holds.

**DEFINITION 3.5.** (The row-version): Let  $\Gamma$  be a  $k \times n$  matrix ( $k$  rows,  $n$  columns), and  $A$  be  $n \times n$  matrix. The pair  $(\Gamma, A)$  is said to be **obstrollable**<sup>11</sup> if the linear span of the set of  $n$ -vector-rows  $\{v \Gamma (\lambda I - A)^{-1}\}$ , where  $v$  runs over the space  $\mathbb{C}^k$  of all  $k$ -vector-rows and  $\lambda$  runs over  $\mathbb{C} \setminus \sigma_A$  ( $\sigma_A$  is the spectrum of  $A$ ), coincides with the whole space  $\mathbb{C}^n$  (of all  $n$ -vector-rows).

The equivalent definitions:

- I. The pair  $(\Gamma, A)$  is said to be **obstrollable**, if the linear span of the set of vectors  $\{v \Gamma A^m\}$ , where  $v$  runs over  $\mathbb{C}^k$  of all  $k$  vector-rows and  $m$  runs over the set  $\mathbb{N}$  of all natural numbers, coincides with the whole space  $\mathbb{C}^n$  (of all  $n$ -vector-rows).
- II. The pair  $(\Gamma, A)$  is said to be **obstrollable**, if the linear span of the set of vectors  $\{v \Gamma \varphi(A)\}$ , where  $v$  runs over  $\mathbb{C}^k$  of all  $k$  vector-rows and  $\varphi$  runs over the set of all functions holomorphic on  $\sigma_A$ , coincides with the whole space  $\mathbb{C}^n$  (of all  $n$ -vector-rows).  $\square$

**DEFINITION 3.5** (The column-version): Let  $\Gamma$  be a  $n \times k$  matrix ( $n$  rows,  $k$  columns), and  $B$  be  $n \times n$  matrix. the pair  $(B, \Gamma)$  is said to be **obstrollable**, if the linear span of the set of  $n$ -vector-columns  $\{(\mu I - B)^{-1} \Gamma v\}$ , where  $v$  runs over the space  $\mathbb{C}^k$  of all  $k$ -vector-columns and  $\mu$  runs over  $\mathbb{C} \setminus \sigma_B$  ( $\sigma_B$  is the spectrum of  $B$ ), coincides with the whole space  $\mathbb{C}^n$  (of all  $n$ -vector-columns).

The equivalent definitions:

- I. The pair  $(B, \Gamma)$  is said to be **obstrollable**, if the linear span of the set of vectors  $\{\Gamma B^m v\}$ , where  $v$  runs over  $\mathbb{C}^k$  of all  $k$  vector-columns and  $m$  runs over the set  $\mathbb{N}$  of all natural numbers, coincides with the whole space  $\mathbb{C}^n$  (of all  $n$ -vector-columns).
- II. The pair  $(B, \Gamma)$  is said to be **obstrollable**, if the linear span of the set of vectors  $\{\varphi(B) \Gamma v\}$ , where  $v$  runs over  $\mathbb{C}^k$  of all  $k$  vector-columns and  $\varphi$  runs over the set of all functions holomorphic on  $\sigma_B$ , coincides with the whole space  $\mathbb{C}^n$  (of all  $n$ -vector-columns).  $\square$

**COMMENT TO TERMINOLOGY:** This terminology is motivated by the system theory. (In

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<sup>11</sup>The word **obstrollable** is a mixture of the words **observable** and **controllable**.

more detail, by the theory of linear time invariant dynamical systems). In this theory structures like  $F(\lambda I - A)^{-1}$  and  $(\mu I - B)^{-1}G$  appear, where  $F$  and  $G$  are  $k \times n$  and  $n \times k$  matrices and  $A, B$  are  $n \times n$  matrix, and usually  $n$  is much bigger than  $k$  ( $n \gg k$ ). If  $F$  is interpreted as the input operator of the system and  $G$  is interpreted as its output operator, then the notions of controllability of the pair  $(F, A)$  and the notion of observability of the pair  $(B, G)$  are introduced. If  $G$  is interpreted as the input operator of the system and  $F$  is interpreted as its output operator, then the notions of controllability of the pair  $(B, G)$  and the notion of observability of the pair  $(F, A)$  are introduced. Structures analogous to the structure  $F(\lambda I - A)^{-1}$ ,  $(\mu I - B)^{-1}G$  appear in the system representation problems as well. We need to formulate the property which is controllability if  $\Gamma$  is interpreted as the input operator, and is observability, if  $\Gamma$  is interpreted as the output operator. However, we would not like to give the preference to one of two: in- or out- interpretations of the matrix  $\Gamma$ . Because this, we choose the “neutral” term *obstrollability*.  $\square$

LEMMA 3.2. Let  $F, G$  be  $k \times n$  and  $n \times k$  matrices, and  $A, B, M$  are  $n \times n$  matrices. Assume that

1.  $F(\lambda I - A)^{-1}M(\mu I - B)^{-1}G \equiv 0 \quad (\forall \lambda \in \mathbb{C} \setminus \sigma_A, \quad \forall \mu \in \mathbb{C} \setminus \sigma_B).$
2. The pairs  $(F, A)$  and  $(B, G)$  are obstrollable.

Then  $M = 0$ .

PROOF. The proof follows immediately from the definition of obstrollability.  $\square$

Thus, the foregoing reasonings (the reasoning of this section and the reasoning used for the first proof of Theorem 2.5) prove the following

THEOREM 3.2. I). Let  $R$  be a rational function such that<sup>12</sup> the chain matrix function  $R(x)(R(y))^{-1}$  admits the representation of the form (2.7) some  $k \times n$  matrix  $F_{\mathcal{P}}$ ,  $n \times k$  matrix  $G_{\mathcal{N}}$  and  $n \times n$  matrices  $A_{\mathcal{P}}, A_{\mathcal{N}}$  and  $H^r$ .

If the pairs  $(F_{\mathcal{P}}, A_{\mathcal{P}})$  and  $(A_{\mathcal{N}}, G_{\mathcal{N}})$  are obstrollable, then the the matrix  $H^r$  satisfy the equality (3.23);

II). Let  $R$  be a rational function such that<sup>12</sup> the chain matrix function  $(R(x))^{-1}R(y)$  admits the representation of the form (2.8) some  $k \times n$  matrix  $F_{\mathcal{N}}$ ,  $n \times k$  matrix  $G_{\mathcal{P}}$  and  $n \times n$  matrices  $A_{\mathcal{P}}, A_{\mathcal{N}}$  and  $H^l$ .

If the pairs  $(F_{\mathcal{N}}, A_{\mathcal{N}})$  and  $(A_{\mathcal{P}}, G_{\mathcal{P}})$  are obstrollable, then the the matrix  $H^l$  satisfy the equality (3.24);

III). Let  $R$  be a rational function such that the chain matrix functions  $R(x)(R(y))^{-1}$  and  $(R(x))^{-1}R(y)$  admit the representations of the form (2.7) and (2.8) respectively, with some  $k \times n$  matrices  $F_{\mathcal{P}}, F_{\mathcal{N}}$ , some  $n \times k$  matrices  $G_{\mathcal{N}}, G_{\mathcal{P}}$  and some  $n \times n$  matrices  $A_{\mathcal{P}}, A_{\mathcal{N}}, H^r, H^l$ .

If all four pairs  $(F_{\mathcal{P}}, A_{\mathcal{P}})$ ,  $(A_{\mathcal{N}}, G_{\mathcal{N}})$ ,  $(F_{\mathcal{N}}, A_{\mathcal{N}})$  and  $(A_{\mathcal{P}}, G_{\mathcal{P}})$  are obstrollable, and if moreover the coupling relations (2.17) hold, then the matrices  $H^r, H^l$  are mutually inverse:  $H^r \cdot H^l = I, H^l \cdot H^r = I$ , and for their inverse matrices  $S^r = (H^r)^{-1}$  and  $S^l = (H^l)^{-1}$  the equalities (2.52), (2.53) holds, i.e. the matrices  $S^r$  and  $S^l$  are solutions of the Sylvester-Lyapunov equations (2.54) and (2.55).

THEOREM 3.3. Let  $R$  be a rational function such that the chain function  $R(x)(R(y))^{-1}$  admits

<sup>12</sup> We assume neither the matrices  $A_{\mathcal{P}}, A_{\mathcal{N}}$  are diagonal, nor their spectra are simple, nor their spectra are non-intersecting. We also don't assume a priori that the matrices  $H^r, H^l$  are invertible.

the representation of the form (2.58) with <sup>13</sup> some  $k \times n$  and  $n \times k$  matrices  $F_{\mathcal{P}}, G_{\mathcal{N}}, n \times n$  matrices  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  and some invertible  $n \times n$  matrix  $S^r$ . If the pairs  $(F_{\mathcal{P}}, A_{\mathcal{P}})$  and  $(A_{\mathcal{N}}, G_{\mathcal{N}})$  are obstrollable, then the equality (2.52) holds, i.e. the matrix  $S^r$  is a solution of the Sylvester-Lyapunov equation (2.54).

The “left” version of this theorem holds as well.

**THEOREM 3.4.** *Let  $R$  be a rational function such that the chain function  $(R(x))^{-1}R(y)$  admits the representation of the form (2.59) with <sup>13</sup> some  $k \times n$  and  $n \times k$  matrices  $F_{\mathcal{N}}, G_{\mathcal{P}}, n \times n$  matrices  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  and some invertible  $n \times n$  matrix  $S^l$ . If the pairs  $(F_{\mathcal{N}}, A_{\mathcal{N}})$  and  $(A_{\mathcal{P}}, G_{\mathcal{P}})$  are obstrollable, then the equality (2.53) holds, i.e. the matrix  $S^l$  is a solution of the Sylvester-Lyapunov equation (2.55).*

**LEMMA 3.3.** *Let  $F = (f_1, \dots, f_n)$  be a  $k \times n$  matrix (i.e.  $f_1, \dots, f_n$  are  $k$  vector-columns), and  $A = \text{diag}(\alpha_1, \dots, \alpha_n)$  be a diagonal matrix with simple spectrum (i.e. all diagonal entries  $\alpha_1, \dots, \alpha_n$  are pairwise different). If no column  $f_1, \dots, f_n$  of the matrix  $F$  is equal to zero, then the pair  $(F, A)$  is obstrollable.*

**PROOF.** Actually, the proof of the statement was already done (See how we obtained the equality (3.20)).  $\square$

This lemma (together with Theorem 2.1) shows that Theorem 3.2 and Theorem 3.3 are applicable to rational matrix functions in general position.

Now we “inverse” our reasonings leading from a chain identity to a Sylvester-Lyapunov equation. Let  $F, G$  be  $k \times n$  and  $n \times k$  matrices,  $A, B$  be  $n \times n$  matrices, with spectra  $\sigma_A$  and  $\sigma_B$ , and  $S$  be an invertible  $n \times n$  matrix for which the equality

$$BS - SA = GF \quad (3.25)$$

holds. Evidently, this equality is equivalent to the identity (with respect to  $y \in \mathbb{C}$ ):

$$S(yI - A) - (yI - B)S \equiv GF.$$

Multiplying the last identity by the matrix  $S^{-1}(yI - B)^{-1}$  from the left and by the matrix  $(yI - A)^{-1}S^{-1}$  from the right, we come to the identity

$$S^{-1}(yI - B)^{-1} - (yI - A)^{-1}S^{-1} - S^{-1}(yI - B)^{-1}G \cdot F(yI - A)^{-1}S^{-1} \equiv 0.$$

(This is nothing more than the equality  $M = 0$ , where  $M$  is defined by (3.17)). Multiplying the last identity by the matrix  $F(xI - A)^{-1}$  from the left and by the matrix  $(zI - B)^{-1}G$  from the right, we come to the identity (with respect to  $x \in \mathbb{C}, y \in \mathbb{C}, z \in \mathbb{C}$ ):

$$\begin{aligned} & F(xI - A)^{-1}S^{-1} \cdot (yI - B)^{-1}(zI - B)^{-1} \cdot G - \\ & - F \cdot (xI - A)^{-1}(yI - A)^{-1} \cdot S^{-1}(zI - B)^{-1}G - \\ & F(xI - A)^{-1}S^{-1}(yI - B)^{-1}G \times \\ & \times F(yI - A)^{-1}S^{-1}(zI - B)^{-1}G \equiv 0. \end{aligned}$$

Using Hilbert identities

$$\begin{aligned} (xI - A)^{-1}(yI - A)^{-1} &\equiv -\frac{(xI - A)^{-1} - (yI - A)^{-1}}{x - y}, \\ (yI - B)^{-1}(zI - B)^{-1} &\equiv -\frac{(yI - B)^{-1} - (zI - B)^{-1}}{y - z}, \end{aligned}$$

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<sup>13</sup>We assume neither the matrices  $A_{\mathcal{P}}, A_{\mathcal{N}}$  are diagonal, nor their spectra are simple, not their spectra are non-intersecting. Of course, the invertibility of the matrix  $S^r$  imposes implicitly some restrictions on the matrices  $A_{\mathcal{P}}, A_{\mathcal{N}}, F_{\mathcal{P}}, G_{\mathcal{N}}$ .

we reduce the last identity to the form

$$\frac{K(x, z) - K(y, z)}{x - y} - \frac{K(x, y) - K(x, z)}{y - z} \equiv K(x, y) \cdot K(y, z),$$

or, what is the same, to the identity

$$(x - y)K(x, y) + (y - z)K(y, z) - (x - z)K(x, z) \equiv -(x - y)(y - z)K(x, z). \quad (3.26)$$

where now the function  $K(\xi, \eta)$  of two variables is defined by the formula

$$K(x, y) \stackrel{\text{def}}{=} F(xI - A)^{-1} \cdot S^{-1} \cdot (yI - B)^{-1}G. \quad (3.27)$$

The identity (3.26) may be rewritten in the form a chain identity (3.1):  $T(x, y) \cdot T(y, z) \equiv T(x, z)$ , where the function  $T(., .)$  of two variables is defined as  $T(x, y) \stackrel{\text{def}}{=} I + (x - y)K(x, y)$ , or

$$T(x, y) \stackrel{\text{def}}{=} I + (x - y)F(xI - A)^{-1}S^{-1}(yI - B)^{-1}G, \quad \mathcal{D}_T \stackrel{\text{def}}{=} (\overline{\mathbb{C}} \setminus \sigma_A) \times (\overline{\mathbb{C}} \setminus \sigma_B).$$

Thus, we proved the following

**THEOREM 3.5.** *Let  $F, G$  be  $k \times n$  and  $n \times k$  matrices,  $A, B$  be  $n \times n$  matrices with spectra  $\sigma_A$  and  $\sigma_B$ , and  $S$  be an invertible  $n \times n$  matrix for which the equality  $BS - SA = GF$  holds.*

*Then the matrix function  $T(., .)$ , which is defined by*

$$T(x, y) \stackrel{\text{def}}{=} I + (x - y)F(xI - A)^{-1}S^{-1}(yI - B)^{-1}G, \quad \mathcal{D}_T \stackrel{\text{def}}{=} (\overline{\mathbb{C}} \setminus \sigma_A) \times (\overline{\mathbb{C}} \setminus \sigma_B), \quad (3.28)$$

*satisfies the chain identity (3.1):*

$$T(x, y) \cdot T(y, z) \equiv T(x, z)$$

*and the diagonal unity identity (3.2):*

$$T(x, x) \equiv I,$$

*and hence<sup>14</sup>, is of form*

$$T(x, y) = R(x)R^{-1}(y),$$

*where  $R(x) \stackrel{\text{def}}{=} T(x, \infty)$ ,  $(R(y))^{-1} \stackrel{\text{def}}{=} T(\infty, y)$ :*

$$R(x) = I - F(xI - A)^{-1}S^{-1}G, \quad R^{-1}(y) = I + FS^{-1}(yI - B)^{-1}G. \quad (3.29)$$

*are mutually inverse (i.e.  $R(x)R^{-1}(x) \equiv R^{-1}(x)R(x) \equiv I$ ) rational matrix functions.*

**PROOF.** The chain identity (3.1) for the function  $T$ , defined by (3.28), was proved immediately before. The diagonal unity identity (3.2) evidently follows from the expression (3.28). The equality  $T(x, y) \equiv T(x, \infty) \cdot T(\infty, y)$  is the special case of the chain identity (3.1) (written for the triple of points  $x, \infty, y$ ). See Theorem 3.1.

Letting  $y$  tend to  $\infty$ , we obtain the expression (3.29) for the matrix function  $R(x) \stackrel{\text{def}}{=} T(x, \infty)$ . Letting  $x$  tend to  $\infty$ , we obtain the expression (3.29) for the matrix function  $R^{-1} \stackrel{\text{def}}{=} T(\infty, y)$ .

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<sup>14</sup>According to Theorem 3.1: the set  $(\overline{\mathbb{C}} \setminus \sigma_A) \cap (\overline{\mathbb{C}} \setminus \sigma_B) = \overline{\mathbb{C}} \setminus (\sigma_A \cup \sigma_B)$  is not only nonempty but also very rich. So, we have many possibilities for choice of a distinguished point. (See the proof of the Theorem 3.1). In particular, we can choose the point  $\infty$  as a distinguished point.

That the functions  $R$  and  $R_{-1}$ , defined by (3.29)), are mutually inverse follows from the chain identity written for the triples  $x, \infty, x$  and  $\infty, x, \infty$ . That the function  $R$  is rational is evident.  $\square$

However, Theorem 3.5 says nothing about the nature of the rational function  $R$ . Imposing restrictions on the data  $A, B, F, G$ , we can say more about the matrix function  $R$ .

**THEOREM 3.6.** *Let  $F$  be  $k \times n$  matrix and  $G$  be  $n \times k$  matrix with non-zero columns and non-zero rows respectively, i.e.*

$$F = [f_1 \ f_2 \ \dots \ f_n], \quad G = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix},$$

where no column  $f_1, f_2, \dots, f_n$  and no row  $g_1, g_2, \dots, g_n$  are zero, and let  $A, B$  be  $n \times n$  be diagonal matrices with simple disjoint spectra, i.e.

$$A = \text{diag}(\lambda_1, \dots, \lambda_n), \quad B = \text{diag}(\mu_1, \dots, \mu_n),$$

where  $\lambda_1, \dots, \lambda_n; \mu_1, \dots, \mu_n$  are pairwise different complex numbers.

Assume that the  $n \times n$  matrix  $S$ ,

$$S = \|s_{p,q}\|_{1 \leq p,q \leq n}, \quad s_{p,q} = \frac{g_p f_q}{\mu_p - \lambda_q} \quad (3.30)$$

(which can be obtained from the data  $F, G, A, B$  as the unique solution of the Sylvester-Lyapunov equation  $BX - XA = GF$ ) is invertible.

Then:

1. The matrix function  $T(\cdot, \cdot)$  of two variables, which is defined by

$$T(x, y) \stackrel{\text{def}}{=} I + (x - y)F(xI - A)^{-1}S^{-1}(yI - B)^{-1}G, \quad \mathcal{D}_T \stackrel{\text{def}}{=} (\overline{\mathbb{C}} \setminus \sigma_A) \times (\overline{\mathbb{C}} \setminus \sigma_B), \quad (3.28)$$

satisfies the chain identity (3.1):

$$T(x, y) \cdot T(y, z) \equiv T(x, z),$$

and the diagonal unity identity (3.2):

$$T(x, x) \equiv I.$$

2. The matrix function  $T(x, y)$  is of the form

$$T(x, y) = R(x)R^{-1}(y),$$

where the matrix functions  $R, R^{-1}$  are defined by the formulas  $R(x) \stackrel{\text{def}}{=} T(x, \infty)$ ,  $R^{-1}(y) \stackrel{\text{def}}{=} T(\infty, y)$ :

$$R(x) = I - F(xI - A)^{-1}S^{-1}G, \quad R^{-1}(y) = I + FS^{-1}(yI - B)^{-1}G. \quad (3.29)$$

and are mutually inverse, (i.e.  $R(x)R^{-1}(x) \equiv R^{-1}(x)R(x) \equiv I$ ).

3. The matrix functions  $R$  and  $R^{-1}$  are rational matrix functions in general position.

4. The pole set  $\mathcal{P}(R)$  of the function  $R$  coincides with the set  $\{\lambda_1, \dots, \lambda_n\}$ ; the zero set  $\mathcal{N}(R)$  of the function  $R$  coincides with the set  $\{\mu_1, \dots, \mu_n\}$ , i.e.

$$A_{\mathcal{P}} = A, \quad A_{\mathcal{N}} = B, \quad (3.31)$$

where  $A_{\mathcal{P}}$  and  $A$  are the pole and zero matrices of the matrix function  $R$ .

5. The semi-residual matrices<sup>15</sup>  $F_{\mathcal{P}}, G_{\mathcal{P}}, F_{\mathcal{N}}, G_{\mathcal{N}}$  of the matrix function  $R$  can be expressed in terms of the data  $F, G$  and of the matrix  $S$  (which in its turn is expressible from the data  $F, G, A, B$ ):

$$F_{\mathcal{P}} = F; \quad G_{\mathcal{N}} = G; \quad F_{\mathcal{N}} = F \cdot S^{-1}; \quad G_{\mathcal{P}} = -S^{-1} \cdot G; \quad (3.32)$$

6. The right zero-pole coupling matrix  $S^r$  and the left zero-pole coupling matrix  $S^l$  for the matrix function  $R$  can be expressed in terms of the matrix  $S$ :

$$S^r = S; \quad S^l = S^{-1}. \quad (3.33)$$

PROOF.

- Items 1 and 2 of Theorem 3.6 are already proved. (See Theorem 3.5).
- Let's prove that the matrix functions  $R$  and  $R^{-1}$  are in general positions and investigate their singularities. The expression (3.29) for  $R$  may be written in the form

$$R(x) = I + \sum_{1 \leq j \leq n} \frac{R_{\lambda_j}}{x - \lambda_j},$$

with the matrix  $R_{\lambda_j}$  is of the form

$$R_{\lambda_j} = f_j \cdot v_j,$$

where  $v_j$  is  $k$ -th row of the  $n \times k$  matrix  $V \stackrel{\text{def}}{=} -S^{-1}G$ . From this expression it follows that  $R$  is holomorphic outside the points  $\{\lambda_1, \dots, \lambda_n\}$  and its inverse  $R^{-1}$  is holomorphic outside the points  $\{\mu_1, \dots, \mu_n\}$ . Let's focus on the point  $\lambda_j$ . There are two possibilities: or  $v_j = 0$ , or  $v_j \neq 0$ . If  $v_j = 0$  then  $R_{\lambda_j} = 0$  and hence, the function  $R$  is holomorphic at the point  $\lambda_j$ . If  $v_j \neq 0$ , then the matrix  $R_{\lambda_j}$  is non-zero, and has rank one. (We recall that, according to the assumptions of Theorem 3.6,  $f_j \neq 0$ .) We show now that the equality  $v_j = 0$  is impossible. This equality may be written in the form  $E_j S^{-1} G = 0$ , where  $E_j = \text{diag}(\delta_{1j}, \delta_{2j}, \dots, \delta_{nj})$ , ( $\delta$  is the Kronecker symbol). Multiplying the identity<sup>16</sup>  $BS - SA = GF$  by the matrix  $E_j S^{-1}$  from the left and by the matrix  $S^{-1}$  from the right and taking into account that the matrices  $e_j$  and  $A$  are permutable (both are diagonal), we come to the equality  $(E_j S^{-1})B - A(E_j S^{-1}) = 0$ . Because the spectra of  $A$  and  $B$  are disjoint, we obtain that  $E_j S^{-1} = 0$ , and hence  $E_j = 0$ . The contradiction shows that the equality  $v_j = 0$  is impossible. Thus, each point  $\lambda_j$ ,  $j = 1, 2, \dots, n$ , is simple pole of the matrix function  $R$ , with residue matrices of rank one.

- Analogously, we can show that the matrix function  $R^{-1}$  is holomorphic outside the points  $\{\mu_1, \dots, \mu_n\}$  and that each point  $\mu_j$ ,  $j = 1, 2, \dots, n$ , is a simple pole of the matrix function  $R^{-1}$ , with residue matrices of rank one. Thus, items 3 and 4 of the claim of Theorem 3.6 are proved.
- Item 5 of the claim follows from the representations (3.29). (Compare (3.29) with (1.37)-(1.38)).

<sup>15</sup>More precisely, one of the representatives of the equivalence class of the set of semi-residual matrices of the matrix function  $R$ . See Remark 1.2.

<sup>16</sup>Which, in fact, serves as the definition of the matrix  $S$ .

• Now that we have established the relation (3.32) we may rewrite the equality  $BS - SA = GF$  in the form  $A_N S - S A_P = G_N F_P$ . Comparing the last equality with equality (2.52), we conclude that  $S^r = S$ . From (2.57) it follows now that  $S^l = S^{-1}$ .  $\square$

The “hybrid” version of this theorem (see Remark 2.9 and formulas (2.66), (2.67)) can be formulated as well. This is the form which is convenient for applications in study of the Schlesinger system.

**THEOREM 3.7.** *Let  $F$  be  $k \times n$  matrix and  $G$  be  $n \times k$  matrix with non-zero columns and non-zero rows respectively, i.e.*

$$F = [f_1 \ f_2 \ \dots \ f_n], \quad G = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix},$$

where no column  $f_1, f_2, \dots, f_n$  and no row  $g_1, g_2, \dots, g_n$  are zero, and let  $A, B$  be  $n \times n$  be diagonal matrices with simple disjoint spectra, i.e

$$A = \text{diag}(\lambda_1, \dots, \lambda_n), \quad B = \text{diag}(\mu_1, \dots, \mu_n),$$

where  $\lambda_1, \dots, \lambda_n; \mu_1, \dots, \mu_n$  are pairwise different complex numbers. Assume that the  $n \times n$  matrix  $S$ ,

$$S = \|s_{p,q}\|_{1 \leq p,q \leq n}, \quad s_{p,q} = \frac{g_p f_q}{\lambda_p - \mu_q} \quad (3.34)$$

(which can be obtained from the data  $F, G, A, B$  as the unique solution of the Sylvester-Lyapunov equation  $AX - XB = GF$ ) is invertible.

Then:

1. The matrix function  $T$  of two variables, which is defined by the formula

$$T(x, y) \stackrel{\text{def}}{=} I - (x - y)FS^{-1}(xI - A)^{-1}S(yI - B)^{-1}S^{-1}G, \quad (3.35)$$

satisfies the chain identity (3.1):

$$T(x, y) \cdot T(y, z) \equiv T(x, z),$$

and the diagonal unity identity (3.2):

$$T(x, x) \equiv I.$$

2. The matrix function  $T(x, y)$  is of the form

$$T(x, y) = R(x)R^{-1}(y),$$

where the matrix functions  $R, R^{-1}$  are defined by the formulas  $R(x) \stackrel{\text{def}}{=} T(x, \infty)$ ,  $R^{-1}(y) \stackrel{\text{def}}{=} T(\infty, y)$ :

$$R(x) = I + FS^{-1}(xI - A)^{-1}G, \quad R^{-1}(y) = I - F(yI - B)^{-1}S^{-1}G. \quad (3.29)$$

and are mutually inverse, (i.e.  $R(x)R^{-1}(x) \equiv R^{-1}(x)R(x) \equiv I$ ).

3. The matrix functions  $R$  and  $R^{-1}$  are rational matrix functions in general position.
4. The pole set  $\mathcal{P}(R)$  of the function  $R$  coincides with the set  $\{\lambda_1, \dots, \lambda_n\}$ ; the zero set  $\mathcal{N}(R)$  of the function  $R$  coincides with the set  $\{\mu_1, \dots, \mu_n\}$ , i.e.

$$A_{\mathcal{P}} = A, \quad A_{\mathcal{N}} = B, \quad (3.36)$$

where  $A_{\mathcal{P}}$  and  $A_{\mathcal{N}}$  are the pole and zero matrices of the matrix function  $R$ .

5. The semi-residual matrices<sup>17</sup>  $F_{\mathcal{P}}, G_{\mathcal{P}}, F_{\mathcal{N}}, G_{\mathcal{N}}$  of the matrix function  $R$  can be expressed in terms of the data  $F, G$  and of the matrix  $S$  (which in its turn is expressible from the data  $F, G, A, B$ ):

$$F_{\mathcal{N}} = F; \quad G_{\mathcal{P}} = G; \quad F_{\mathcal{P}} = F \cdot S^{-1}; \quad G_{\mathcal{N}} = -S^{-1} \cdot G; \quad (3.37)$$

6. The right zero-pole coupling matrix  $S^r$  and the left zero-pole coupling matrix  $S^l$  for the matrix function  $R$  can be expressed in terms of the matrix  $S$ :

$$S^l = S; \quad S^r = S^{-1}. \quad (3.38)$$

PROOF. Theorem 3.7 is nothing more then Theorem 3.6 in “other variables”. Let  $A, B, F, G$  be the data of Theorem 3.7, and  $S$  be the matrix (3.34) generated by this data. Let’s introduce the matrices

$$\tilde{F} = FS^{-1}, \quad \tilde{G} = -S^{-1}G, \quad \tilde{S} = S^{-1}.$$

Equality  $AS - SB = GF$  rewritten in terms of  $A, B, \tilde{F}, \tilde{G}, \tilde{S}$  becomes the form  $B\tilde{S} - \tilde{S}A = \tilde{G}\tilde{F}$ . From the last equality it is easy to see that no column of the matrix  $\tilde{F}$  and no row of the matrix  $\tilde{G}$  equals zero: the equality  $E_j\tilde{G} = 0$  or  $\tilde{F}E_j = 0$ , where  $E_j = \text{diag}(\delta_{1j}, \delta_{2j}, \dots, \delta_{nj})$ , ( $\delta$  is the Kronecker symbol), leads to the equality  $E_j\tilde{S} = 0$  or  $\tilde{S}E_j = 0$ , what contradicts to the invertibility of  $\tilde{S}$ . Now Theorem 3.6, applied to the matrix function

$$T(x, y) = I + (x - y)\tilde{F}(xI - A)^{-1}(\tilde{S})^{-1}(yI - B)^{-1}\tilde{G},$$

gives the chain and diagonal unity identities for this  $T$  and the factorization  $T(x, y) = R(x)R^{-1}(y)$ , as well as the expressions for the semi-residual matrices  $F_{\mathcal{P}}, G_{\mathcal{P}}, F_{\mathcal{N}}, G_{\mathcal{N}}$  of the matrix function  $R$ :

$$F_{\mathcal{P}} = \tilde{F}, \quad G_{\mathcal{N}} = \tilde{G}, \quad F_{\mathcal{N}} = \tilde{F} \cdot \tilde{S}^{-1}, \quad G_{\mathcal{P}} = -\tilde{S}^{-1}\tilde{G}.$$

Rewritten in terms of  $F, G, S$ , these relations becomes the form (3.37). □

#### 4. THE SYSTEM REPRESENTATION AS A TOOL FOR THE SPECTRAL (WIENER-HOPF) FACTORIZATION OF MATRIX FUNCTIONS.

In this section we show that the system representation may be used as an efficient tool for the so called spectral factorization (or the Wiener-Hopf factorization) of a matrix function. The problem of the spectral factorization can be formulated in the following way.

GEOMETRIC CONFIGURATION. In the extended complex plane  $\overline{\mathbb{C}}$  a simple closed contour  $\Gamma$  is given. This contour separates  $\overline{\mathbb{C}}$  into two regions,  $G_+$  and  $G_-$ . These regions  $G_+$  and  $G_-$  are connected open sets. We assume that the point  $\{\infty\}$  does not belong to the contour  $\Gamma$ , thus one of the components, say  $G_-$ , contains the point  $\{\infty\}$ .

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<sup>17</sup>See the footnote <sup>15</sup>.



DEFINITION 4.1. Given a  $k \times k$  matrix function  $\Phi$  on the contour  $\Gamma$ , the factorization of the form

$$\Phi(\zeta) = \Phi_+(\zeta) \cdot \Phi_-(\zeta) \quad (\zeta \in \Gamma), \quad (4.1)$$

where  $\Phi_+$  and  $\Phi_-$  are  $k \times k$  matrix functions, the matrix function  $\Phi_+$  and its inverse  $(\Phi_+)^{-1}$  are holomorphic on  $G_- \cup \Gamma$ , and the matrix function  $\Phi_-$  and its inverse  $(\Phi_-)^{-1}$  are holomorphic<sup>18</sup> on  $G_+ \cup \Gamma$ , is said to be the spectral factorization (or the Wiener-Hopf factorization) of the matrix-function  $\Phi$  with respect to  $\Gamma$ .  $\square$

We impose the normalizing condition

$$\Phi_+(\infty) = I. \quad (4.2)$$

on the factor  $\Phi_+$ . (The function  $\Phi_+$  is holomorphic and invertible at the point  $\infty$ , so the condition (4.2) makes sense).

Under the normalizing condition (4.2), the spectral factorization (4.1) is unique.

Even in the scalar case  $k = 1$  (i.e.  $\Phi$  is a complex valued function) the factorization problem (4.1) is not always solvable: there is a topological obstacle for the solvability. For a smooth nonvanishing complex valued function  $\Phi$  on  $\Gamma$ , the factorization problem (4.1) solvable if and only if there exists an univalued continuous branch of the function  $\ln \Phi(\zeta)$  on  $\Gamma$ . If this condition is fulfilled, the solution of the factorization problem may be expressed in terms of the data  $\Phi$  by the formula

$$\Phi_{\pm}(\zeta) = \exp \left\{ \pm \frac{1}{2\pi i} \int_{\Gamma} \frac{\ln \Phi(t)}{t - \zeta} dt \right\} \quad (z \in G_{\pm}). \quad (4.3)$$

The proof of the fact, that the formula (4.3) gives the solution of the factorization problem (4.1) in the scalar case, is based essentially on the Sokhotskiĭ-Plemelj formulas on the boundary behavior of the Cauchy integral. Actually, in the scalar case we solve the additive problem

$$\Psi_+(\zeta) + \Psi_-(\zeta) = \ln \Phi(\zeta) \quad (\zeta \in \Gamma),$$

and then we exponentiate. In the matricial case we still can solve the appropriate additive problem using the Cauchy integral, but exponentiating does not lead to the desirable result: In view of noncommutativity of the matricial multiplication,  $\exp\{A + B\} \neq \exp\{A\} \cdot \exp\{B\}$  for matrices  $A$  and  $B$  in general. In the matricial case, the situation with the factorization problem (4.1) is much more complicated than in the scalar case. There are not only topological obstacles to the solvability of this problem. The factorization problem (4.1) is equivalent to some system of singular integral equations on  $\Gamma$ , and to analyze this system is approximately so hard as to investigate the original factorization problem (4.1). The factorization problem (4.1) appeared firstly in the context of Hilbert's twenty-first problem: to construct a Fuchsian linear differential system with the prescribed monodromy group. See [Pl2], [Bo] and [Gah] for details and historical references. The factorization of the type (4.1) is used also in the solving systems of singular integrals equations with Cauchy kernel on the contour as well as for solving of systems of integral equations which kernel depends on the differences of the arguments on the half axis. See [Vek], [GoKr] and [ClGo] on this subject. It should be mentioned that the pioneer papers [Bir1] and [Bir2] by G. Birkhoff had a profound impact on the further investigations on matrix factorization.

We confine ourself to the case the function  $\Phi$  is a rational matrix function (or, more precisely, the restriction on  $\Gamma$  of a rational matrix function) such that the functions  $\Phi$  and  $\Phi^{-1}$  are

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<sup>18</sup>In particular, the functions  $\Phi_+$  and  $\Phi_-$  are holomorphic on the common boundary  $\Gamma$  of the domains  $G_+$  and  $G_-$ , so the relation (4.1) makes sense.

holomorphic on the contour  $\Gamma$ . In this case the factorization (4.1) is **global**, i.e. the matrix functions  $\Phi_+$  and  $\Phi_-$  are rational, and the equality

$$\Phi(z) = \Phi_+(z) \cdot \Phi_-(z) \quad (\forall z \in \overline{\mathbb{C}}) \quad (4.4)$$

holds. Indeed, in this case the function  $\Phi_+^{-1}\Phi$  is holomorphic within  $G_-$  except finite many poles located on the set  $\mathcal{P}(\Phi) \cap G_-$ . In view of (4.1), this function continues analytically into the function  $\Phi_-$  which is holomorphic on  $G_+ \cup \Gamma$ . Thus, the matrix-function  $\Phi_-$  has no other singularities in  $\overline{\mathbb{C}}$  than finite many poles and hence is rational. For the same reasoning, the matrix function  $\Phi_+$  is rational.

Thus, in the case that the initial matrix function  $\Phi$  is rational, the problem of the spectral factorization may be reformulated in the following manner:

**DEFINITION 4.1'.** *Given a  $k \times k$  rational matrix function  $\Phi$ ,  $\det \Phi \not\equiv 0$ , its factorization of the form (4.4), where  $\Phi_+$ ,  $\Phi_-$  are rational matrix functions with zero and pole location*

$$\mathcal{P}(\Phi_+) \subset G_+, \quad \mathcal{N}(\Phi_+) \subset G_+, \quad \mathcal{P}(\Phi_-) \subset G_-, \quad \mathcal{N}(\Phi_-) \subset G_-, \quad (4.5)$$

*is said to be the spectral factorization (or the Wiener-Hopf factorization) of the matrix-function  $\Phi$  with respect to  $\Gamma$ .*  $\square$

We consider even the more special case: the function  $\Phi$  is a rational function in general position. In this case the calculation of the factors  $\Phi_+$  and  $\Phi_-$  can be performed completely by hand, in terms of poles and “zeros” of the matrix function  $\Phi$  and its semiresidual vectors.

So, let  $\Phi$  be a rational matrix function in general position, normalized by the condition

$$\Phi(\infty) = I. \quad (4.6)$$

Let  $\mathcal{P}(\Phi)$  and  $\mathcal{N}(\Phi)$  be its pole and zero sets,  $A_{\mathcal{P}}(\Phi)$  and  $A_{\mathcal{N}}(\Phi)$  be its pole and zero matrices,  $F_{\mathcal{P}}(\Phi)$ ,  $G_{\mathcal{P}}(\Phi)$ ,  $F_{\mathcal{N}}(\Phi)$ ,  $G_{\mathcal{N}}(\Phi)$  be the appropriate semiresidual matrices. According to the Theorem 2.6, the zero-pole coupling matrices  $S^r(\Phi)$  and  $S^l(\Phi)$  are invertible, and the matrix-function  $\Phi$  admits the representations of the form (2.60) and (2.62):

$$\Phi(z) = I - F_{\mathcal{P}}(\Phi)(zI - A_{\mathcal{P}}(\Phi))^{-1}S^r(\Phi)^{-1}G_{\mathcal{N}}(\Phi) \quad (4.7)$$

and

$$\Phi(z) = I + F_{\mathcal{N}}(\Phi)S^l(\Phi)^{-1}(zI - A_{\mathcal{P}}(\Phi))^{-1}G_{\mathcal{P}}(\Phi). \quad (4.8)$$

The Sylvester-Lyapunov equations for the matrices  $S^r(\Phi)$  and  $S^l(\Phi)$  (which actual-y are the definitions of these matrices) are of the form:

$$A_{\mathcal{N}}(\Phi)S^r(\Phi) - S^r(\Phi)A_{\mathcal{P}}(\Phi) = G_{\mathcal{N}}(\Phi)F_{\mathcal{P}}(\Phi), \quad (4.9)$$

and

$$A_{\mathcal{P}}(\Phi)S^l(\Phi) - S^l(\Phi)A_{\mathcal{N}}(\Phi) = G_{\mathcal{P}}(\Phi)F_{\mathcal{N}}(\Phi). \quad (4.10)$$

Moreover, the matrices  $S^r(\Phi)$  and  $S^l(\Phi)$  satisfy the equality

$$S^l(\Phi) \cdot S^r(\Phi) = I. \quad (4.11)$$

(This is (2.57) for the matrix function  $\Phi$ ). The zero-pole coupling relation hold:

$$\begin{aligned} \text{a). } G_{\mathcal{N}}(\Phi) &= -S^r(\Phi)G_{\mathcal{P}}(\Phi); & \text{b). } G_{\mathcal{P}}(\Phi) &= -S^l(\Phi)G_{\mathcal{N}}(\Phi); \\ \text{c). } F_{\mathcal{P}}(\Phi) &= F_{\mathcal{N}}(\Phi)S^r(\Phi); & \text{d). } F_{\mathcal{N}}(\Phi) &= F_{\mathcal{P}}(\Phi)S^l(\Phi). \end{aligned} \quad (4.12)$$

(This is (2.51) for the matrix-function  $\Phi$ ).

According to the assumptions,

$$\text{a). } \mathcal{P}(\Phi) \cap \Gamma = \emptyset; \quad \text{b). } \mathcal{N}(\Phi) \cap \Gamma = \emptyset. \quad (4.13)$$

Assume that the factorization (4.4) holds, and that the normalizing condition (4.2) is satisfied.

Since the functions  $\Phi_-$ ,  $\Phi_-^{-1}$  are holomorphic in  $G_+$  and the functions  $\Phi$ ,  $\Phi^{-1}$  have only simple poles in  $G_+$ , from the relations

$$\text{a). } \Phi_+ = \Phi \cdot \Phi_-^{-1}, \quad \text{b). } \Phi_+^{-1} = \Phi_- \cdot \Phi^{-1} \quad (4.14)$$

it follows that the functions  $\Phi_+$ ,  $\Phi_+^{-1}$  have only simple poles in  $G_+$ . In  $G_-$  and on  $\Gamma$  the functions  $\Phi_+$  and  $\Phi_+^{-1}$  don't have singularities at all. Thus,

$$\mathcal{P}(\Phi_+) = \mathcal{P}(\Phi) \cap G_+, \quad \mathcal{N}(\Phi_+) = \mathcal{N}(\Phi) \cap G_+. \quad (4.15)$$

Let  $\lambda \in \mathcal{P}(\Phi_+)$ . From (4.14.a) it follows that the residues  $R_\lambda(\Phi_+)$  and  $R_\lambda(\Phi)$  of the matrix functions  $\Phi_+$  and  $\Phi$  at the point  $\lambda$  are related by the equality

$$R_\lambda(\Phi_+) = R_\lambda(\Phi) \cdot (\Phi_-(\lambda))^{-1} \quad (\forall \lambda \in \mathcal{P}(\Phi_+)). \quad (4.16)$$

Since  $\Phi$  is a matrix function in general position, the rank of the residue matrix  $R_\lambda(\Phi)$  is equal to one. According to the assumptions, the matrix  $\Phi_-(\lambda)$  is invertible for  $\lambda \in G_+$ . From (4.16) it follows now that the rank of the residue matrix  $R_\lambda(\Phi_+)$  is equal to one as well. Let now  $\mu \in \mathcal{N}(\Phi_+)$ . From (4.14.b) it follows that the residues  $R_\mu(\Phi_+)$  and  $R_\mu(\Phi)$  of the matrix functions  $\Phi_+^{-1}$  and  $\Phi^{-1}$  at the point  $\mu$  are related by the equality

$$R_\mu(\Phi_+) = \Phi_-(\mu) \cdot R_\mu(\Phi) \quad (\forall \mu \in \mathcal{N}(\Phi_+)). \quad (4.17)$$

The rank of the matrix  $R_\mu(\Phi)$  is equal to one ( $\Phi$  is a matrix function in general position); the matrix  $\Phi_-(\mu)$  is invertible (according to the assumptions, the matrix  $\Phi_-(\mu)$  is invertible for  $\mu \in G_+$ ). From (4.17) it follows now, that the rank of the residue matrix  $R_\mu(\Phi_+)$  is equal to one as well. From (4.15) it follows that

$$\mathcal{P}(\Phi_+) \cap \mathcal{N}(\Phi_+) = \emptyset; \quad \{\infty\} \notin \mathcal{P}(\Phi_+), \quad \{\infty\} \notin \mathcal{N}(\Phi_+).$$

Thus,  $\Phi_+$  is a rational matrix function in general position.

In the same way we obtain that  $\Phi_-$  is a rational matrix function in general position, and

$$\mathcal{P}(\Phi_-) = \mathcal{P}(\Phi) \cap G_-, \quad \mathcal{N}(\Phi_-) = \mathcal{N}(\Phi) \cap G_-. \quad (4.18)$$

Moreover, the residues  $R_\lambda(\Phi_-)$  and  $R_\mu(\Phi_-)$  of the matrix functions  $\Phi_-$  and  $\Phi_-^{-1}$  at the poles  $\lambda \in \mathcal{P}(\Phi_-)$  and  $\mu \in \mathcal{N}(\Phi_-)$  ( $= \mathcal{P}(\Phi_-^{-1})$ ) are related to the residues of the matrix functions  $\Phi$  and  $\Phi^{-1}$  at the same points by the equalities

$$R_\lambda(\Phi_-) = (\Phi_+(\lambda))^{-1} \cdot R_\lambda(\Phi) \quad (\forall \lambda \in \mathcal{P}(\Phi_-)) \quad (4.19)$$

and

$$R_\mu(\Phi_-) = R_\mu(\Phi) \cdot \Phi_+(\mu) \quad (\forall \mu \in \mathcal{N}(\Phi_-)). \quad (4.20)$$

From (4.16) and (4.17) it follows that the left semiresidual vectors of the matrix functions  $\Phi_+$  and  $\Phi$  at the poles  $\lambda \in \mathcal{P}(\Phi_+)$  coincide:

$$f_\lambda(\Phi_+) = f_\lambda(\Phi) \quad (\forall \lambda \in \mathcal{P}(\Phi_+)). \quad (4.21)$$

and the right semiresidual vectors of the matrix functions  $\Phi_+^{-1}$  and  $\Phi^{-1}$  at the poles  $\mu \in \mathcal{N}(\Phi_+)$  coincide:

$$g_\mu(\Phi_+^{-1}) = g_\mu(\Phi^{-1}) \quad (\forall \mu \in \mathcal{N}(\Phi_+)). \quad (4.22)$$

In the same way we can obtain that the right semiresidual vectors of the matrix functions  $\Phi_-$  and  $\Phi$  coincide:

$$g_\lambda(\Phi_-) = g_\lambda(\Phi) \quad (\forall \lambda \in \mathcal{P}(\Phi_-)). \quad (4.23)$$

and the left semiresidual vectors of the matrix functions  $\Phi_-^{-1}$  and  $\Phi^{-1}$  coincide:

$$f_\mu(\Phi_-^{-1}) = f_\mu(\Phi^{-1}) \quad (\forall \mu \in \mathcal{N}(\Phi_-)). \quad (4.24)$$

The equalities (4.21) – (4.24) are crucial for solving of the considered factorization problem.

According to Lemma 1.2, the equality

$$\#\mathcal{P}(\Phi_+) = \#\mathcal{N}(\Phi_+)$$

holds for the rational matrix function in general position  $\Phi_+$ . Taking into account the equality (4.15), we obtain the following equality

$$\#(\mathcal{P}(\Phi) \cap G_+) = \#(\mathcal{N}(\Phi) \cap G_+) \stackrel{\text{def}}{=} n_+. \quad (4.25)$$

Of course, the equality

$$\#(\mathcal{P}(\Phi) \cap G_-) = \#(\mathcal{N}(\Phi) \cap G_-) \stackrel{\text{def}}{=} n_-. \quad (4.26)$$

holds as well.

To simplify notations, we denote

$$\mathcal{P}_+ \stackrel{\text{def}}{=} \mathcal{P}(\Phi) \cap G_+; \quad \mathcal{P}_- \stackrel{\text{def}}{=} \mathcal{P}(\Phi) \cap G_-; \quad \mathcal{N}_+ \stackrel{\text{def}}{=} \mathcal{N}(\Phi) \cap G_+; \quad \mathcal{N}_- \stackrel{\text{def}}{=} \mathcal{N}(\Phi) \cap G_-. \quad (4.27)$$

To the decompositions

$$\begin{aligned} \mathcal{P}(\Phi) &= \mathcal{P}_+ \cup \mathcal{P}_-, \quad \mathcal{P}_+ \cap \mathcal{P}_- = \emptyset, \\ \mathcal{N}(\Phi) &= \mathcal{N}_+ \cup \mathcal{N}_-, \quad \mathcal{N}_+ \cap \mathcal{N}_- = \emptyset \end{aligned}$$

of the pole and zero sets  $\mathcal{P}(\Phi)$ ,  $\mathcal{N}(\Phi)$  of the matrix function  $\Phi$  there correspond natural block-decompositions of the matrices which appear in the system representations of the matrix functions  $\Phi$  and  $\Phi^{-1}$ : pole and zero matrices  $A_{\mathcal{P}}(\Phi)$ ,  $A_{\mathcal{N}}(\Phi)$ , the semiresidual matrices  $F_{\mathcal{P}}(\Phi)$ ,  $F_{\mathcal{N}}(\Phi)$ ,  $G_{\mathcal{P}}(\Phi)$ ,  $G_{\mathcal{N}}(\Phi)$  as well as the zero-pole coupling matrices  $S^r(\Phi)$  and  $S^l(\Phi)$ .

Namely, the decompositions of the pole and zero matrices  $A_{\mathcal{P}}(\Phi) = \text{diag}(\lambda_l)_{\lambda_l \in \mathcal{P}(\Phi)}$  and  $A_{\mathcal{N}}(\Phi) = \text{diag}(\mu_l)_{\mu_l \in \mathcal{N}(\Phi)}$  (of the dimension  $n \times n$ ) are of the form:

$$A_{\mathcal{P}}(\Phi) = \begin{bmatrix} A_{\mathcal{P}}(\Phi)_1 & 0 \\ 0 & A_{\mathcal{P}}(\Phi)_2 \end{bmatrix}, \quad (4.28)$$

$$A_{\mathcal{N}}(\Phi) = \begin{bmatrix} A_{\mathcal{N}}(\Phi)_1 & 0 \\ 0 & A_{\mathcal{N}}(\Phi)_2 \end{bmatrix}, \quad (4.29)$$

where

$$A_{\mathcal{P}}(\Phi)_1 = \text{diag}(\lambda_l)_{\lambda_l \in \mathcal{P}_+}; \quad A_{\mathcal{P}}(\Phi)_2 = \text{diag}(\lambda_l)_{\lambda_l \in \mathcal{P}_-};$$

$$A_{\mathcal{N}}(\Phi)_1 = \text{diag}(\mu_l)_{\mu_l \in \mathcal{N}_+}; \quad A_{\mathcal{N}}(\Phi)_2 = \text{diag}(\mu_l)_{\mu_l \in \mathcal{N}_-};$$

$A_{\mathcal{P}}(\Phi)_1$  and  $A_{\mathcal{N}}(\Phi)_1$  are diagonal  $n_+ \times n_+$  matrices,  $A_{\mathcal{P}}(\Phi)_2$  and  $A_{\mathcal{N}}(\Phi)_2$  are diagonal  $n_- \times n_-$  matrices; ( $n_+$  and  $n_-$  are defined in (4.25) and (4.26):  $n_+ = \#\mathcal{P}_+ = \#\mathcal{N}_+$ ;  $n_- = \#\mathcal{P}_- = \#\mathcal{N}_-$ ).

The block-decompositions of the semiresidual matrices  $F_{\mathcal{P}}(\Phi), F_{\mathcal{N}}(\Phi), G_{\mathcal{P}}(\Phi), G_{\mathcal{N}}(\Phi)$ :

$$\begin{aligned} F_{\mathcal{P}}(\Phi) &= \text{row}(f_{\lambda_l})_{\lambda_l \in \mathcal{P}(\Phi)}, \quad F_{\mathcal{N}}(\Phi) = \text{row}(f_{\mu_l})_{\mu_l \in \mathcal{N}(\Phi)} \quad \text{are } k \times n \text{ matrices,} \\ G_{\mathcal{P}}(\Phi) &= \text{col}(g_{\lambda_l})_{\lambda_l \in \mathcal{P}(\Phi)}, \quad G_{\mathcal{N}}(\Phi) = \text{col}(g_{\mu_l})_{\mu_l \in \mathcal{N}(\Phi)} \quad \text{are } n \times k \text{ matrices,} \end{aligned}$$

related to the matrix-function  $\Phi$ , are of the form:

$$\text{p). } F_{\mathcal{P}}(\Phi) = [F_{\mathcal{P}}(\Phi)_1 \ F_{\mathcal{P}}(\Phi)_2]; \quad \text{n). } F_{\mathcal{N}}(\Phi) = [F_{\mathcal{N}}(\Phi)_1 \ F_{\mathcal{N}}(\Phi)_2]. \quad (4.30)$$

and

$$\text{p). } G_{\mathcal{P}}(\Phi) = \begin{bmatrix} G_{\mathcal{P}}(\Phi)_1 \\ G_{\mathcal{P}}(\Phi)_2 \end{bmatrix}; \quad \text{n). } G_{\mathcal{N}}(\Phi) = \begin{bmatrix} G_{\mathcal{N}}(\Phi)_1 \\ G_{\mathcal{N}}(\Phi)_2 \end{bmatrix}, \quad (4.31)$$

where

$$\begin{aligned} F_{\mathcal{P}}(\Phi)_1 &= \text{row}(f_{\lambda_l})_{\lambda_l \in \mathcal{P}_+}, \quad F_{\mathcal{N}}(\Phi)_1 = \text{row}(f_{\mu_l})_{\mu_l \in \mathcal{N}_+} \quad \text{are } k \times n_+ \text{ matrices;} \\ F_{\mathcal{P}}(\Phi)_2 &= \text{row}(f_{\lambda_l})_{\lambda_l \in \mathcal{P}_-}, \quad F_{\mathcal{N}}(\Phi)_2 = \text{row}(f_{\mu_l})_{\mu_l \in \mathcal{N}_-} \quad \text{are } k \times n_- \text{ matrices;} \\ G_{\mathcal{P}}(\Phi)_1 &= \text{col}(g_{\lambda_l})_{\lambda_l \in \mathcal{P}_+}, \quad G_{\mathcal{N}}(\Phi)_1 = \text{col}(g_{\mu_l})_{\mu_l \in \mathcal{N}_+} \quad \text{are } n_+ \times k \text{ matrices;} \\ G_{\mathcal{P}}(\Phi)_2 &= \text{col}(g_{\lambda_l})_{\lambda_l \in \mathcal{P}_-}, \quad G_{\mathcal{N}}(\Phi)_2 = \text{col}(g_{\mu_l})_{\mu_l \in \mathcal{N}_-} \quad \text{are } n_- \times k \text{ matrices.} \end{aligned}$$

From (4.15) and (4.18) it follows that

$$\text{p). } A_{\mathcal{P}}(\Phi_+) = A_{\mathcal{P}}(\Phi)_1 \quad \text{n). } A_{\mathcal{N}}(\Phi_+) = A_{\mathcal{N}}(\Phi)_1 \quad (4.32)$$

and

$$\text{p). } A_{\mathcal{P}}(\Phi_-) = A_{\mathcal{P}}(\Phi)_2 \quad \text{n). } A_{\mathcal{N}}(\Phi_-) = A_{\mathcal{N}}(\Phi)_2. \quad (4.33)$$

In view of (4.21) and (4.22),

$$\text{p). } F_{\mathcal{P}}(\Phi_+) = F_{\mathcal{P}}(\Phi)_1, \quad \text{n). } G_{\mathcal{N}}(\Phi_+) = G_{\mathcal{N}}(\Phi)_1. \quad (4.34)$$

In view of (4.23) and (4.24),

$$\text{p). } F_{\mathcal{N}}(\Phi_-) = F_{\mathcal{N}}(\Phi)_2, \quad \text{n). } G_{\mathcal{P}}(\Phi_-) = G_{\mathcal{P}}(\Phi)_2. \quad (4.35)$$

Thus, we have expressed the pole matrix  $A_{\mathcal{P}}(\Phi_+)$  and semiresidual matrices  $F_{\mathcal{P}}(\Phi_+), G_{\mathcal{N}}(\Phi_+)$  for the left factor  $\Phi_+$  in terms of blocks of the appropriate block-decompositions of the pole matrix  $A_{\mathcal{P}}(\Phi)$  and semiresidual matrices  $F_{\mathcal{P}}(\Phi), G_{\mathcal{N}}(\Phi)$  for the factorized matrix function  $\Phi$ . We also have expressed the zero matrix  $A_{\mathcal{N}}(\Phi_-)$  and semiresidual matrices  $F_{\mathcal{N}}(\Phi_-), G_{\mathcal{P}}(\Phi_-)$  for the right factor  $\Phi_-$  in terms of blocks of the matrices  $A_{\mathcal{N}}(\Phi), F_{\mathcal{N}}(\Phi), G_{\mathcal{P}}(\Phi)$ . In principle, these data are sufficient to recover the factors  $\Phi_+$  and  $\Phi_-$  (from the appropriate blocks of the pole and semiresidual matrices for the factored matrix function  $\Phi$ ). To carry out the recovering, we have to solve the Sylvester-Lyapunov equations (2.52) and (2.53) for the matrix functions  $\Phi_+$  and  $\Phi_-$  respectively to find from these equations the zero-pole coupling matrices  $S^r(\Phi_+)$  and  $S^l(\Phi_-)$ :

$$A_{\mathcal{N}}(\Phi_+)S^r(\Phi_+) - S^r(\Phi_+)A_{\mathcal{P}}(\Phi_+) = G_{\mathcal{N}}(\Phi_+)F_{\mathcal{P}}(\Phi_+), \quad (4.36)$$

$$A_{\mathcal{P}}(\Phi_-)S^l(\Phi_-) - S^l(\Phi_-)A_{\mathcal{P}}(\Phi_-) = G_{\mathcal{P}}(\Phi_-)F_{\mathcal{N}}(\Phi_-). \quad (4.37)$$

According to the assertion 1 of Theorem 2.6 (actually, according to Theorem 2.5: see (2.31)), their solutions  $S^r(\Phi_+)$  and  $S^r(\Phi_-)$  are invertible matrices. Then we construct the factors  $\Phi_+$  and  $\Phi_-$  according to the formulas (2.60) and (2.62):

$$\Phi_+(z) = I - F_{\mathcal{P}}(\Phi_+)(zI - A_{\mathcal{P}}(\Phi_+))^{-1}(S^r(\Phi_+))^{-1}G_{\mathcal{N}}(\Phi_+) \quad (4.38)$$

and

$$\Phi_-(z) = I + F_{\mathcal{N}}(\Phi_-)(S^l(\Phi_-))^{-1}(zI - A_{\mathcal{P}}(\Phi_-))^{-1}G_{\mathcal{P}}(\Phi_-). \quad (4.39)$$

We express now the matrices  $S^r(\Phi_+)$  and  $S^l(\Phi_-)$  in terms of blocks of the matrices  $S^r(\Phi)$  and  $S^l(\Phi)$ . The block-decomposition of the zero-pole coupling matrices  $S^r(\Phi)$  and  $S^l(\Phi)$  are of the form:

$$S^r(\Phi) = \begin{bmatrix} S^r(\Phi)_{11} & S^r(\Phi)_{12} \\ S^r(\Phi)_{21} & S^r(\Phi)_{22} \end{bmatrix}, \quad (4.40)$$

$$S^l(\Phi) = \begin{bmatrix} S^l(\Phi)_{11} & S^l(\Phi)_{12} \\ S^l(\Phi)_{21} & S^l(\Phi)_{22} \end{bmatrix}, \quad (4.41)$$

where 11- block-entries  $S^r(\Phi)_{11}$ ,  $S^l(\Phi)_{11}$  are  $n_+ \times n_+$  matrices, and 22- block-entries  $S^r(\Phi)_{22}$ ,  $S^l(\Phi)_{22}$  are  $n_- \times n_-$  matrices ( $n_+$ ,  $n_-$  are defined in (4.25) and (4.26)). The block-decompositions (4.40) and (4.41) are consistent with the block-decompositions (4.28)–(4.31) of the pole and semiresidual matrices. The Sylvester-Lyapunov equation (4.9) for the matrix  $S^r(\Phi)$ , written in the block-matrical form

$$\begin{aligned} \begin{bmatrix} A_{\mathcal{N}}(\Phi)_1 & 0 \\ 0 & A_{\mathcal{N}}(\Phi)_2 \end{bmatrix} \begin{bmatrix} S^r(\Phi)_{11} & S^r(\Phi)_{12} \\ S^r(\Phi)_{21} & S^r(\Phi)_{22} \end{bmatrix} - \begin{bmatrix} S^r(\Phi)_{11} & S^r(\Phi)_{12} \\ S^r(\Phi)_{21} & S^r(\Phi)_{22} \end{bmatrix} \begin{bmatrix} A_{\mathcal{P}}(\Phi)_1 & 0 \\ 0 & A_{\mathcal{P}}(\Phi)_2 \end{bmatrix} \\ = \begin{bmatrix} G_{\mathcal{N}}(\Phi)_1 \\ G_{\mathcal{N}}(\Phi)_2 \end{bmatrix} [F_{\mathcal{P}}(\Phi)_1 \ F_{\mathcal{P}}(\Phi)_2], \end{aligned}$$

may be considered as a system of matricial equations for the block-entries of the matrix  $S^r(\Phi)$ . This system is decomposed into four equations for block-entries of this matrix.

In particular, the equation for the entry  $S^r(\Phi)_{11}$  is of the form

$$A_{\mathcal{N}}(\Phi)_1 S^r(\Phi)_{11} - S^r(\Phi)_{11} A_{\mathcal{P}}(\Phi)_1 = G_{\mathcal{N}}(\Phi)_1 F_{\mathcal{P}}(\Phi)_1. \quad (4.42)$$

The Sylvester-Lyapunov equation (4.10) for the matrix  $S^l(\Phi)$ , written in the block-matrical form, may be considered as a system of matricial equations for the block-entries of the matrix  $S^l(\Phi)$ . In particular, the equation for the entry  $S^l(\Phi)_{22}$  is of the form

$$A_{\mathcal{P}}(\Phi)_2 S^l(\Phi)_{22} - S^l(\Phi)_{22} A_{\mathcal{N}}(\Phi)_2 = G_{\mathcal{P}}(\Phi)_2 F_{\mathcal{N}}(\Phi)_2. \quad (4.43)$$

We show that

$$S^r(\Phi_+) = S^r(\Phi)_{11} \quad (4.44)$$

and

$$S^l(\Phi_-) = S^l(\Phi)_{22}. \quad (4.45)$$

The easiest way to do this is to use the explicit formulas (2.56) for the matrices  $S^r$ ,  $S^l$  in terms of the poles, zeros and semiresidual matrices of the matrix function. For the function  $\Phi$  the formula (2.56.r) takes the form

$$S^r(\Phi) = \|s_{p,q}^r(\Phi)\|_{\mu_p \in \mathcal{N}(\Phi), \lambda_q \in \mathcal{P}(\Phi)}, \quad s_{p,q}^r(\Phi) = \frac{g_{\mu_p}(\Phi) \cdot f_{\lambda_q}(\Phi)}{\mu_p - \lambda_q}. \quad (4.46)$$

In particular,

$$S^r(\Phi)_{11} = \|s_{p,q}^r(\Phi)\|_{\mu_p \in \mathcal{N}(\Phi) \cap G_+, \lambda_q \in \mathcal{P}(\Phi) \cap G_+}, \quad s_{p,q}^r(\Phi) = \frac{g_{\mu_p}(\Phi) \cdot f_{\lambda_q}(\Phi)}{\mu_p - \lambda_q}. \quad (4.47)$$

For the function  $\Phi_+$  the formula (2.56.r) takes the form

$$S^r(\Phi_+) = \|s_{p,q}^r(\Phi_+)\|_{\mu_p \in \mathcal{N}(\Phi_+), \lambda_q \in \mathcal{P}(\Phi_+)}, \quad s_{p,q}^r(\Phi_+) = \frac{g_{\mu_p}(\Phi_+) \cdot f_{\lambda_q}(\Phi_+)}{\mu_p - \lambda_q} \quad (4.48)$$

Comparing two last formulas and taking into account (4.15), (4.21) and (4.22), we conclude that (4.44) holds. In the same way, comparing the formulas

$$S^l(\Phi) = \|s_{p,q}^l(\Phi)\|_{\lambda_p \in \mathcal{P}(\Phi), \mu_q \in \mathcal{N}(\Phi)}, \quad s_{p,q}^l(\Phi) = \frac{g_{\lambda_p}(\Phi) \cdot f_{\mu_q}(\Phi)}{\lambda_p - \mu_q} \quad (4.49)$$

and

$$S^l(\Phi_-) = \|s_{p,q}^l(\Phi_-)\|_{\lambda_p \in \mathcal{P}(\Phi_-), \mu_q \in \mathcal{N}(\Phi_-)}, \quad s_{p,q}^l(\Phi_-) = \frac{g_{\lambda_p}(\Phi_-) \cdot f_{\mu_q}(\Phi_-)}{\lambda_p - \mu_q} \quad (4.50)$$

and taking into account (4.18), (4.23) and (4.24), we conclude that (4.45) holds.

The explicit expressions (4.48), (4.50) for the zero-pole coupling matrices are consequence of the Sylvester-Lyapunov equalities (4.36), (4.37). It is also possible to derive the equations (4.44) and (4.45) directly from the equalities (4.36), (4.37), bypassing the explicit expressions (4.48), (4.50). (4.46), (4.49). The latter way is better, because it is applicable not only to matrix functions in general position.

Taking into account the equalities (4.32), and (4.34), we came from (4.36) to the equation

$$A_{\mathcal{N}}(\Phi)_1 S^r(\Phi_+) - S^r(\Phi_+) A_{\mathcal{P}}(\Phi)_1 = G_{\mathcal{N}}(\Phi)_1 F_{\mathcal{P}}(\Phi)_1. \quad (4.51)$$

Comparing (4.42) and (4.51) and taking into account the uniqueness of the solution of the Sylvester-Lyapunov equation, we conclude that the equality (4.44) holds.

In the same way, we can establish the equality (4.45). Taking into account the equalities (4.33), (4.35), we came from (4.37) to the equation

$$A_{\mathcal{P}}(\Phi)_2 S^l(\Phi_-) - S^l(\Phi_-) A_{\mathcal{N}}(\Phi)_2 = G_{\mathcal{P}}(\Phi)_2 F_{\mathcal{N}}(\Phi)_2. \quad (4.52)$$

Comparing (4.43) and (4.52) and taking into account the uniqueness of the solution of the Sylvester-Lyapunov equation, we conclude that the equality (4.45) holds.

According to Theorem 2.6 (applied to the rational matrix functions in general position  $\Phi_+$  and  $\Phi_-$ ), the matrices  $S^r(\Phi_+)$  and  $S^l(\Phi_-)$  are invertible. In view of (4.44) and (4.45), the block-entries  $(S^r(\Phi))_{11}$  and  $(S^l(\Phi))_{22}$  (of the matrices  $S^r(\Phi)$  and  $S^l(\Phi)$  respectively) are invertible.

Now we may rewrite the formulas (4.38), (4.39) for the factors  $\Phi_+$  and  $\Phi_-$  in terms of block-entries of the pole-, zero-, semiresidual- and zero-pole coupling matrices for the factorized matrix function  $\Phi$ . Substituting the expressions (4.32.p), (4.34) and (4.44) into (4.38), we obtain

$$\Phi_+(z) = I - F_{\mathcal{P}}(\Phi)_1 (zI - A_{\mathcal{P}}(\Phi)_1)^{-1} (S^r(\Phi)_{11})^{-1} G_{\mathcal{N}}(\Phi)_1. \quad (4.53)$$

Analogously, substituting the expressions (4.32.n), (4.35) and (4.45) into (4.39), we obtain

$$\Phi_-(z) = I + F_{\mathcal{N}}(\Phi)_2 (S^l(\Phi)_{22})^{-1} (zI - A_{\mathcal{P}}(\Phi)_2)^{-1} G_{\mathcal{P}}(\Phi)_2. \quad (4.54)$$

The formula (4.53) expresses the factor  $\Phi_+$  in terms of the values  $F_{\mathcal{P}}(\Phi)$ ,  $G_{\mathcal{N}}(\Phi)$  and  $S^r(\Phi)$ , which appear in the representation (4.7), whereas the formula (4.54) expresses the factor  $\Phi_-$  in terms of the values  $F_{\mathcal{N}}(\Phi)$ ,  $G_{\mathcal{P}}(\Phi)$  and  $S^l(\Phi)$ , which appear in the other representation (4.8). This disagreement is inconvenient for some calculations. Therefore we also give the formula which expresses the factor  $\Phi_-$  in terms of the the same values  $F_{\mathcal{P}}(\Phi)$ ,  $G_{\mathcal{N}}(\Phi)$  and  $S^r(\Phi)$ , which appear in the expression (4.53) for the factor  $\Phi_+$ . To do it, we have to use the zero-pole coupling relations (4.12.a), (4.12.c), Together with (4.35.p) and (4.34.n), these relations mean:

$$F_{\mathcal{N}}(\Phi_-) = (F_{\mathcal{P}}(\Phi) S^r(\Phi)^{-1})_2; \quad G_{\mathcal{P}}(\Phi_-) = -((S^r(\Phi))^{-1} G_{\mathcal{N}}(\Phi))_2. \quad (4.55)$$

$\Phi$ ): The equality (4.11) together with (4.45) means:

$$S^l(\Phi_-) = (S^r(\Phi)^{-1})_{22}. \quad (4.56)$$

Substituting the expressions (4.55) and (4.56) into (4.39), we obtain

$$\Phi_-(z) = I - \left( F_{\mathcal{P}}(\Phi) S^r(\Phi)^{-1} \right)_2 \left( \left( S^r(\Phi)^{-1} \right)_{22} \right)^{-1} \left( zI - A_{\mathcal{P}}(\Phi)_2 \right)^{-1} \left( S^r(\Phi)^{-1} G_{\mathcal{N}}(\Phi) \right)_2. \quad (4.57)$$

The last formula already expresses the factor  $\Phi_-$  in terms of the values  $F_{\mathcal{P}}(\Phi)$ ,  $G_{\mathcal{N}}(\Phi)$  and  $S^r(\Phi)$ , i.e. in terms of the same values which appear in the representation (4.53) of the factor  $\Phi_+$ .

However, for the further considerations it will be useful to transform this formula, substituting into it the expression for the inverse matrix  $S^r(\Phi)^{-1}$  in terms of the block-entries of the matrix  $S^r(\Phi)$  itself. First of all we recall a formula for the inversion of a  $2 \times 2$  block matrix with square diagonal block-entries. Let

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \quad (4.58)$$

be a square  $n \times n$  block-matrix matrix, which block-entries  $m_{11}$  and  $m_{22}$  be square  $n_1 \times n_1$  and  $n_2 \times n_2$  matrices respectively ( $n = n_1 + n_2$ ). We assume that the matrix  $m_{11}$  is invertible. To inverse the matrix  $M$ , we first of all factorize it:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ m_{21} m_{11}^{-1} & I \end{bmatrix} \cdot \begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} - m_{21} m_{11}^{-1} m_{12} \end{bmatrix} \cdot \begin{bmatrix} I & m_{11}^{-1} m_{12} \\ 0 & I \end{bmatrix}. \quad (4.59)$$

The matrices  $\begin{bmatrix} I & 0 \\ m_{21} m_{11}^{-1} & I \end{bmatrix}$  and  $\begin{bmatrix} I & m_{11}^{-1} m_{12} \\ 0 & I \end{bmatrix}$  are invertible, and

$$\begin{bmatrix} I & 0 \\ m_{21} m_{11}^{-1} & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -m_{21} m_{11}^{-1} & I \end{bmatrix}; \quad \begin{bmatrix} I & m_{11}^{-1} m_{12} \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -m_{11}^{-1} m_{12} \\ 0 & I \end{bmatrix}$$

Therefore the matrix  $M$  is invertible if and only if the matrix  $\begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} - m_{21} m_{11}^{-1} m_{12} \end{bmatrix}$  is invertible. The latter is invertible if and only if the matrix  $(m_{22} - m_{21} m_{11}^{-1} m_{12})$  is invertible. If the matrix  $(m_{22} - m_{21} m_{11}^{-1} m_{12})$  is invertible <sup>19</sup>, then

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}^{-1} = \begin{bmatrix} I & -m_{11}^{-1} m_{12} \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} m_{11}^{-1} & 0 \\ 0 & (m_{22} - m_{21} m_{11}^{-1} m_{12})^{-1} \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ -m_{21} m_{11}^{-1} & I \end{bmatrix}.$$

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<sup>19</sup>The invertibility of the matrix  $m_{11}$  was assumed from the very beginning



Finally, for the matrix  $M$  with the decomposition (4.58),

$$M^{-1} = \begin{bmatrix} m_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -m_{11}^{-1}m_{12} \\ I \end{bmatrix} \cdot [m_{22} - m_{21} m_{11}^{-1} m_{12}]^{-1} \cdot \begin{bmatrix} -m_{21} m_{11}^{-1} & I \end{bmatrix}. \quad (4.60)$$

In particular,

$$(M^{-1})_{22} = (m_{22} - m_{21} m_{11}^{-1} m_{12})^{-1}, \quad (4.61)$$

If  $F$  is  $k \times n$  matrix, with the block-decomposition  $F = [F_1 \ F_2]$ , where  $F_1, F_2$  are  $k \times n_1$  and  $k \times n_2$  matrices respectively, then 2-entry of the matrix  $F M^{-1}$  is of the form

$$(F M^{-1})_2 = [F_1 \ F_2] \cdot \begin{bmatrix} -m_{11}^{-1}m_{12} \\ I \end{bmatrix} \cdot (m_{22} - m_{21} m_{11}^{-1} m_{12})^{-1}. \quad (4.62)$$

If  $G$  is  $n \times k$  matrix, with the block-decomposition  $G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$ , where  $G_1, G_2$  are  $n_1 \times k$  and  $n_2 \times k$  matrices respectively, then 2-entry of the matrix  $M^{-1} G$  is of the form

$$(M^{-1}G)_2 = (m_{22} - m_{21} m_{11}^{-1} m_{12})^{-1} \cdot [-m_{21} m_{11}^{-1} \ I] \cdot \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}. \quad (4.63)$$

Let us take the matrix  $S^r(\Phi)$  with the decomposition (4.40) as the matrix  $M$  ( $n_1 = n_+, n_2 = n_-$ ) as the matrix  $M$ . Both the matrix  $S^r(\Phi)$  and its block-entry the matrix  $S^r(\Phi)_{11}$  are invertible. Therefore the entry  $(S^r(\Phi)^{-1})_{22}$  of the inverse matrix  $(S^r(\Phi))^{-1} = S^l(\Phi)$  is invertible as well, and, according to (4.61),

$$((S^r(\Phi)^{-1})_{22})^{-1} = S^r(\Phi)_{22} - S^r(\Phi)_{21} S^r(\Phi)_{11}^{-1} S^r(\Phi)_{12}. \quad (4.64)$$

In particular, the matrix  $S^r(\Phi)_{22} - S^r(\Phi)_{21} S^r(\Phi)_{11}^{-1} S^r(\Phi)_{12}$  is invertible. In view of (4.11), the equality (4.45) can be presented in the form

$$S^l(\Phi_-)_{22} = (S^r(\Phi)_{22} - S^r(\Phi)_{21} S^r(\Phi)_{11}^{-1} S^r(\Phi)_{12})^{-1}. \quad (4.65)$$

Taking the matrix  $F_{\mathcal{P}}(\Phi)$  with the decomposition (4.30.p) as the matrix  $F$  and the matrix  $G_{\mathcal{N}}(\Phi)$  with the decomposition (4.31.n) as the matrix  $G$ . we reduce the formula (4.57) to the form

$$\begin{aligned} \Phi_-(z) &= I - [F_{\mathcal{N}}(\Phi)_1 \ F_{\mathcal{N}}(\Phi)_2] \cdot \begin{bmatrix} -S^r(\Phi)_{11}^{-1} S^r(\Phi)_{12} \\ I \end{bmatrix} \cdot (zI - A_{\mathcal{P}}(\Phi)_2)^{-1} \cdot \\ &\cdot (S^r(\Phi)_{22} - S^r(\Phi)_{21} S^r(\Phi)_{11}^{-1} S^r(\Phi)_{12})^{-1} \cdot [-S^r(\Phi)_{21} S^r(\Phi)_{11}^{-1} \ I] \begin{bmatrix} G_{\mathcal{N}}(\Phi)_1 \\ G_{\mathcal{N}}(\Phi)_2 \end{bmatrix}. \end{aligned} \quad (4.66)$$

Thus, we proved the following

**THEOREM 4.1.** *Let  $\Phi$  be a rational  $k \times k$  matrix function in general position, satisfying the normalizing condition (4.6):  $\Phi(\infty) = I$ , with pole and zero sets  $\mathcal{P}(\Phi)$  and  $\mathcal{N}(\Phi)$ , pole and zero matrices  $A_{\mathcal{P}}(\Phi)$  and  $A_{\mathcal{N}}(\Phi)$ , semiresidual matrices  $F_{\mathcal{P}}(\Phi)$ ,  $F_{\mathcal{N}}(\Phi)$ ,  $G_{\mathcal{P}}(\Phi)$ ,  $G_{\mathcal{N}}(\Phi)$ , and zero-pole coupling matrices  $S^r(\Phi)$  and  $S^l(\Phi)$ , which are decomposed into blocks as described above.*

Assume that the function  $\Phi$  admits the spectral factorization with respect to  $\Gamma$ , i.e.

$$\Phi(z) = \Phi_+(z) \cdot \Phi_-(z) \quad (z \in \overline{\mathbb{C}}), \quad (4.4)$$

where  $\Phi_+$ ,  $\Phi_-$  are rational matrix functions, with the pole- and zero-location:

$$\mathcal{P}(\Phi_+) \subset G_+, \quad \mathcal{N}(\Phi_+) \subset G_+, \quad \mathcal{P}(\Phi_-) \subset G_-, \quad \mathcal{N}(\Phi_-) \subset G_-. \quad (4.5)$$

Let the normalizing condition (4.2):  $\Phi_+(\infty) = I$  hold.

Then:

1. For the matrix function  $\Phi$ , the number of poles and the number of “zeros”, located in  $G_+$ , are equal:  $\#(\mathcal{P}(\Phi) \cap G_+) = \#(\mathcal{N}(\Phi) \cap G_+)$ ; the number of poles and the number of “zeros”, located in  $G_-$ , are also equal:  $\#(\mathcal{P}(\Phi) \cap G_-) = \#(\mathcal{N}(\Phi) \cap G_-)$ .
2. The block-entries  $S^r(\Phi)_{11}$  and  $S^l(\Phi)_{22}$  (in the above described block decompositions (4.40), (4.41) of the matrices  $S^r(\Phi)$  and  $S^l(\Phi)$  respectively), as well as the matrix  $S^r(\Phi)_{22} - S^r(\Phi)_{21}S^r(\Phi)_{11}^{-1}S^r(\Phi)_{12}$  are square invertible matrices.
3. The factors  $\Phi_+$  and  $\Phi_-$  are rational matrix functions in general position, which are representable in the form (4.53) and (4.54) respectively.
4. The representation (4.53) is the right system representation of the factor  $\Phi_+$ ; the entries of this representation is expressible in terms of the entries of the right system representation for  $\Phi$ : (4.15), (4.32), (4.34) and (4.44) hold. The representation (4.54) is the left system representation of the factor  $\Phi_-$ ; the entries of this representation is expressible in terms of the entries of the right system representation for  $\Phi$ : (4.18), (4.33), (4.35) and (4.45) hold. The representation (4.54) may be rewritten in terms of the entries of the right system representation for  $\Phi_+$ : (4.66) holds.

The converse statement is true as well.

**THEOREM 4.2.** *Let  $\Phi$  be a rational  $k \times k$  matrix function in general position, satisfying the normalizing condition (4.6):  $\Phi(\infty) = I$ , with pole and zero sets  $\mathcal{P}(\Phi)$  and  $\mathcal{N}(\Phi)$ , pole and zero matrices  $A_{\mathcal{P}}(\Phi)$  and  $A_{\mathcal{N}}(\Phi)$ , semiresidual matrices  $F_{\mathcal{P}}(\Phi)$ ,  $F_{\mathcal{N}}(\Phi)$ ,  $G_{\mathcal{P}}(\Phi)$ ,  $G_{\mathcal{N}}(\Phi)$ , and zero-pole coupling matrices  $S^r(\Phi)$  and  $S^l(\Phi)$ , which are decomposed into blocks as described above.*

Assume that the following conditions are satisfied:

1. None of the poles of  $\Phi$  as well as none of its “zeros” belongs to  $\Gamma$ :

$$\mathcal{P}(\Phi) \cap \Gamma = \emptyset, \quad \mathcal{P}(\Phi^{-1}) \cap \Gamma = \emptyset.$$

The number of poles and the number of zeros of the matrix-function  $\Phi$ , located in  $G_+$ , are equal:

$$\#(\mathcal{P}(\Phi) \cap G_+) = \#(\mathcal{N}(\Phi) \cap G_+),$$

or, what is the same, the number of poles and the number of zeros of the matrix-function  $\Phi$ , located in  $G_-$ , are equal:

$$\#(\mathcal{P}(\Phi) \cap G_-) = \#(\mathcal{N}(\Phi) \cap G_-).$$

2. The block-entry  $S^r(\Phi)_{11}$  of the matrix  $S^r(\Phi)$  (in the above described block decompositions (4.40)) is invertible matrix.

Then:

1. The matrix functions  $\Phi_+$  and  $\Phi_-$ , which are defined by (4.53) and<sup>20</sup> (4.54), are rational matrix functions in general position. For these functions  $\Phi_+$  and  $\Phi_-$ , the conditions (4.5) are satisfied.
2. The matrix function  $\Phi$  admits the spectral factorization (4.4) with respect to  $\Gamma$ , with these matrix-functions  $\Phi_+$  and  $\Phi_-$  as the factors.

PROOF. To investigate the properties of the matrix functions  $\Phi_+$ ,  $\Phi_-$ , we will use Theorem 3.6. To this theorem be applicable to the functions  $\Phi_+$ ,  $\Phi_-$ , we have to check whether its assumptions follow from the assumptions of Theorem 4.2. First of all, we have to be sure that none of the columns of the matrices  $F_{\mathcal{P}}(\Phi)_1$ ,  $F_{\mathcal{N}}(\Phi)_2$  is a zero-column and none of the rows of the matrices  $G_{\mathcal{N}}(\Phi)_1$ ,  $G_{\mathcal{P}}(\Phi)_2$  is a zero-row. This property holds indeed because it holds for the including matrices  $F_{\mathcal{P}}(\Phi)$ ,  $F_{\mathcal{N}}(\Phi)$ ,  $G_{\mathcal{P}}(\Phi)$ ,  $G_{\mathcal{N}}(\Phi)$  (as for semiresidual matrices for the rational matrix function  $\Phi$  in general position). Then we have to check that the matrices  $S^r(\Phi)_{11}$ , which appears at the “core” of the representation (4.53), satisfy some Sylvester-Lyapunov equality of the form  $B S^r(\Phi)_{11} - S^r(\Phi)_{11} A = G_{\mathcal{N}}(\Phi)_1 \cdot F_{\mathcal{P}}(\Phi)_1$ , where  $A = A_{\mathcal{P}}(\Phi)_1$  and  $B$  is a diagonal matrix. In the considered case, the identity (4.42) plays the role of such Sylvester-Lyapunov equality, with  $B = A_{\mathcal{N}}(\Phi)_1$ . As we are already established, the equality (4.42) is the consequence of the equality (4.9). The latter holds as the Sylvester-Lyapunov identity for the matrix function  $\Phi$ . Thus, the representation (4.53) of the function  $\Phi_+$  is its **system** representation. According to Theorem 3.6, the matrix-functions  $\Phi_+$  is a rational matrix-function in general position, which pole- and zero-sets are totalities of the diagonal entries of the matrices  $A_{\mathcal{P}}(\Phi)_1$  and  $A_{\mathcal{N}}(\Phi)_1$  respectively:  $\mathcal{P}(\Phi_+) = \left\{ \text{diag } A_{\mathcal{P}}(\Phi)_1 \right\}$ , and  $\mathcal{N}(\Phi_+) = \left\{ \text{diag } A_{\mathcal{N}}(\Phi)_1 \right\}$ . Hence (see the definition (4.28), (4.29) of the matrices  $A(\mathcal{P})_1$ ,  $A(\mathcal{N})_1$ ), the conditions  $\mathcal{P}(\Phi_+) = \mathcal{P}(\Phi) \cap G_+$ ,  $\mathcal{N}(\Phi_+) = \mathcal{N}(\Phi) \cap G_+$  hold.

The matrix  $S^r(\Phi)$  is invertible as the Sylvester-Lyapunov matrix related to the matrix-function  $\Phi$  (Theorem 2.5. See (4.11)). The corner block-entry  $S^r(\Phi)_{11}$  is invertible by the assumptions of Theorem. Hence (see the identity (4.60) with  $S^r(\Phi)$  as  $M$ ), the matrix  $S^r(\Phi)_{22} - S^r(\Phi)_{21} S^r(\Phi)_{21}^{-1} S^r(\Phi)_{12}$  is invertible as well, and (4.64) holds. In view of (4.11), (4.65) holds. In particular, the block-entry  $S^l(\Phi)_{22}$  is invertible as well. Thus, the matrix-function  $\Phi_-$  is well defined by the formula (4.54). The Sylvester-Lyapunov equality (4.43) for the matrix  $S^l(\Phi)_{22}$  follows from the equality (4.10). (The latter holds as the ‘left’ Sylvester-Lyapunov equality for  $\Phi$ ). According to Theorem 3.6 (to be more precise, according to the “left” version of this Theorem),  $\Phi_-$  is a rational matrix function in general position, which pole- and zero-sets are totalities of the diagonal entries of the matrices  $A_{\mathcal{P}}(\Phi)_2$  and  $A_{\mathcal{N}}(\Phi)_2$  respectively:  $\mathcal{P}(\Phi_-) = \left\{ \text{diag } A_{\mathcal{P}}(\Phi)_2 \right\}$ , and  $\mathcal{N}(\Phi_-) = \left\{ \text{diag } A_{\mathcal{N}}(\Phi)_2 \right\}$ . Hence, the conditions  $\mathcal{P}(\Phi_-) = \mathcal{P}(\Phi) \cap G_-$ ,  $\mathcal{N}(\Phi_-) = \mathcal{N}(\Phi) \cap G_-$  hold. All the more, the conditions (4.5) hold.

Thus, if the equality (4.4) holds for  $\Phi_+$ ,  $\Phi_-$  defined by (4.53), (4.65), it gives the spectral factorization of  $\Phi$ .

One remains only to verify the equality (4.4). The representation (4.54) is convenient to investigate the properties of  $\Phi_-$ . However, to verify the equality (4.4), it is more convenient to use the representation (4.66). To derive (4.66) from (4.54), we have to use the zero-pole coupling relations (4.12), the formula (4.60) for inversion of  $2 \times 2$  block- matrix (applied to the matrix  $S^r(\Phi)$  with the block-decomposition (4.40)) and, in particular, the equality (4.65).

<sup>20</sup>If the block-entry  $S^r(\Phi)_{11}$  is invertible, the block-entry  $S^l(\Phi)_{22}$  is invertible as well.

To abbreviate the notation, we omit some notation entries, like indices etc. So, for example, we write  $S$  instead  $S^r(\Phi)$ ,  $S_{11}$  instead  $S^r(\Phi)_{11}$ ,  $F = [F_1 \ F_2]$  instead (4.30.p),  $G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$  instead (4.30.n),  $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$  instead (4.28),  $B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$  instead (4.29).

Using the  $2 \times 2$ -block-matrix inversion rule (4.60), we present the representation (4.7) for  $\Phi$  in the form

$$\Phi(z) = I - F (zI - A)^{-1} \left( \begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} -S_{11}^{-1} S_{12} \\ I \end{bmatrix} \Delta^{-1} \begin{bmatrix} -S_{21} S_{11}^{-1} & I \end{bmatrix} \right) G, \quad (4.67)$$

where

$$\Delta = S_{22} - S_{21} S_{11}^{-1} S_{12}.$$

The representation (4.53) for the function  $\Phi_+$  we present in the form

$$\Phi_+(z) = I - F(zI - A)^{-1} \begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} G. \quad (4.68)$$

The representation (4.54) for  $\Phi_-$ , rewritten in the form (4.66), is:

$$\Phi_-(z) = I - F \begin{bmatrix} -S_{11}^{-1} S_{12} \\ I \end{bmatrix} (zI - A_2)^{-1} \Delta^{-1} \begin{bmatrix} -S_{21} S_{11}^{-1} & I \end{bmatrix} G. \quad (4.69)$$

Multiplying the expressions in the right hand sides of (4.68) and (4.69) term by term, we obtain:

$$\begin{aligned} \Phi_+(z)\Phi_-(z) &= I - \\ &- F(zI - A)^{-1} \begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} G - \quad (II) \\ &- F \begin{bmatrix} -S_{11}^{-1} S_{12} \\ I \end{bmatrix} (zI - A_2)^{-1} \Delta^{-1} \begin{bmatrix} -S_{21} S_{11}^{-1} & I \end{bmatrix} G + \quad (III) \\ &+ F(zI - A)^{-1} \begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} G \cdot F \begin{bmatrix} -S_{11}^{-1} S_{12} \\ I \end{bmatrix} (zI - A_2)^{-1} \Delta^{-1} \begin{bmatrix} -S_{21} S_{11}^{-1} & I \end{bmatrix}. \quad (IV) \end{aligned} \quad (4.70)$$

Substituting the expression for  $GF$  from the Sylvester-Lyapunov identity  $GF = BS - SA$ , or

$$GF = S(zI - A) - (zI - B)S,$$

into the expression (IV), we obtain:

$$\begin{aligned} (IV) &= \\ &F(zI - A)^{-1} \begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \cdot S(zI - A) \cdot \begin{bmatrix} -S_{11}^{-1} S_{12} \\ I \end{bmatrix} (zI - A_2)^{-1} \Delta^{-1} \begin{bmatrix} -S_{21} S_{11}^{-1} & I \end{bmatrix} - \\ &- F(zI - A)^{-1} \begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \cdot (zI - B)S \cdot \begin{bmatrix} -S_{11}^{-1} S_{12} \\ I \end{bmatrix} (zI - A_2)^{-1} \Delta^{-1} \begin{bmatrix} -S_{21} S_{11}^{-1} & I \end{bmatrix} \end{aligned} \quad (4.71)$$

It is clear that

$$\begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} (zI - B)^{-1} = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}, \quad S \begin{bmatrix} -S_{11}^{-1}S_{12} \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ * \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} (zI - B)^{-1} S \begin{bmatrix} -S_{11}^{-1}S_{12} \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

the second summand in the expression (4.71) for (IV) vanishes, and

$$(IV) = F(zI - A)^{-1} \begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \cdot S(zI - A) \cdot \begin{bmatrix} -S_{11}^{-1}S_{12} \\ I \end{bmatrix} (zI - A_2)^{-1} \Delta^{-1} \begin{bmatrix} -S_{21}S_{11}^{-1} & I \end{bmatrix} G. \quad (4.72)$$

As

$$\begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \cdot S = I - \begin{bmatrix} 0 & -S_{11}^{-1}S_{12} \\ 0 & I \end{bmatrix},$$

$$\begin{aligned} (IV) &= F \begin{bmatrix} -S_{11}^{-1}S_{12} \\ I \end{bmatrix} (zI - A_2)^{-1} \Delta^{-1} \begin{bmatrix} -S_{21}S_{11}^{-1} & I \end{bmatrix} G - \\ &- F(zI - A)^{-1} \begin{bmatrix} 0 & -S_{11}^{-1}S_{12} \\ 0 & I \end{bmatrix} (zI - A) \begin{bmatrix} -S_{11}^{-1}S_{12} \\ I \end{bmatrix} (zI - A_2)^{-1} \Delta^{-1} \begin{bmatrix} -S_{21}S_{11}^{-1} & I \end{bmatrix} G. \end{aligned}$$

It is clear that

$$(zI - A) \begin{bmatrix} -S_{11}^{-1}S_{12} \\ I \end{bmatrix} (zI - A_2)^{-1} = \begin{bmatrix} * \\ I \end{bmatrix},$$

hence

$$\begin{bmatrix} 0 & -S_{11}^{-1}S_{12} \\ 0 & I \end{bmatrix} (zI - A) \begin{bmatrix} -S_{11}^{-1}S_{12} \\ I \end{bmatrix} (zI - A_2)^{-1} = \begin{bmatrix} -S_{11}^{-1}S_{12} \\ I \end{bmatrix}.$$

Thus,

$$\begin{aligned} (IV) &= F \begin{bmatrix} -S_{11}^{-1}S_{12} \\ I \end{bmatrix} (zI - A_2)^{-1} \Delta^{-1} \begin{bmatrix} -S_{21}S_{11}^{-1} & I \end{bmatrix} G - \\ &- F(zI - A)^{-1} \begin{bmatrix} -S_{11}^{-1}S_{12} \\ I \end{bmatrix} \Delta^{-1} \begin{bmatrix} -S_{21}S_{11}^{-1} & I \end{bmatrix} G. \end{aligned} \quad (4.73)$$

From (4.70) and (4.73) it follows that

$$\Phi_+(z) \Phi_-(z) = I - F(zI - A)^{-1} \left( \begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -S_{11}^{-1}S_{12} \\ I \end{bmatrix} \Delta^{-1} \begin{bmatrix} -S_{21}S_{11}^{-1} & I \end{bmatrix} \right) G.$$

Comparing the last expression to (4.67), we conclude that the factorization equality (4.4) holds. Theorem 4.2 is proved.  $\square$

REMARK 4.1. Let  $\Phi$  be a rational matrix function in general position, satisfying the normalizing condition (4.6):  $\Phi(\infty) = I$ . Assume that none of the poles of  $\Phi$  and none of the “zeros” of

$\Phi$  belongs to the contour  $\Gamma$ , and the conditions (4.25) :  $\#(\mathcal{P}(\Phi) \cap G_+) = \#(\mathcal{N}(\Phi) \cap G_+)$  is satisfied; (or, what is the same, the condition (4.26):  $\#(\mathcal{P}(\Phi) \cap G_-) = \#(\mathcal{N}(\Phi) \cap G_-)$  is satisfied). This means that the matrices  $S^r(\Phi)_{11}$  and  $S^l(\Phi)_{22}$  are square one. According to Theorems 4.1 and 4.2, the matrix function  $\Phi$  admits the spectral factorization with respect to  $\Gamma$  if and only if the matrix  $S^r(\Phi)_{11}$  is invertible, or, what is the same, the matrix  $S^l(\Phi)_{22}$  is invertible. What can we say if the condition of the invertibility of the matrix  $S^r(\Phi)_{11}$  is violated? According to the matrix factorization theory, the matrix function  $\Phi$  admits the factorization of the form

$$\Phi(z) = \Phi_+(z) D(z) \Phi_-(z), \quad (4.74)$$

where the matrix functions  $\Phi_+$  and  $\Phi_+^{-1}$  are holomorphic on the set  $G_- \cup \Gamma$ , the matrix functions  $\Phi_-$  and  $\Phi_-^{-1}$  are holomorphic on the set  $G_+ \cup \Gamma$  and  $D(z)$  is the matrix function of the form

$$D(z) = \text{diag} \left( (z - z_0)^{\kappa_1}, (z - z_0)^{\kappa_2}, \dots, (z - z_0)^{\kappa_k} \right), \quad (4.75)$$

$\kappa_1, \kappa_2, \dots, \kappa_k$  are integer numbers. The numbers  $\kappa_1, \kappa_2, \dots, \kappa_k$  are said to be the partial indices of the matrix function  $\Phi$  with respect to the contour  $\Gamma$ .

The point  $z_0$  is an arbitrary chosen fixed point from  $G_+$ . Of course, the factors  $\Phi_+$ ,  $\Phi_-$  and  $D$  from the factorization (4.74) depend of the choice of the point  $z_0$ . However, the set  $\{\kappa_1, \kappa_2, \dots, \kappa_k\}$  (and under the normalizing condition

$$\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_k \quad (4.76)$$

the matrix  $D$ ) is determined uniquely by the function  $\Phi$ . In particular, the partial indices do not depend on the choice of the distinguished point  $z_0$ . In contrast to this, even for the given distinguished point  $z_0$  the matrices  $\Phi_+$  and  $\Phi_-$  are determined non-uniquely. However, this non-uniqueness can be easily described. (See Theorem 7.1 from [GoKr]).

The partial indices play a fundamental role; in the homogeneous Hilbert problem they were first introduced by N.I. Muskhelishvili and N.P. Vekua [MuVe]. (The factorization of another form:  $\Phi(z) = \Phi_+(z) \Phi_-(z) D(z)$  was considered by G. Birkhoff much earlier, in 1913 (see [Bir1]). The matrices  $\Phi_+$ ,  $\Phi_-$  and  $D$  have the same properties that in above described factorization (4.74) (in particular, the matrix  $D$  is of the form (4.75)), but the matrices themselves are different. In particular, Birkhoff's partial indices may be different. The relation between these kinds of factorizations is studied in [FM].

The natural question arise how to compute the factors  $\Phi_+$ ,  $\Phi_-$ ,  $D$  in terms of the pole, zero and semiresidual matrices for the given rational matrix function  $\Phi$  in general position? Of course, there are general methods for performing such a factorization (see, for example, [ClGo]). However, we believe that for a rational matrix function in general position the factorization may be done much more explicitly.  $\square$

**REMARK 4.2.** Let us assume now that the rational matrix function  $\Phi$  in general position depends holomorphically on some parameter  $\alpha \in \mathbb{C}$  and satisfy the normalizing condition (4.6):  $\Phi(\infty, \alpha) \equiv I$ . The factors  $\Phi_+$ ,  $\Phi_-$ ,  $D$  in the factorization (4.74):

$$\Phi(z, \alpha) = \Phi_+(z, \alpha) D(z, \alpha) \Phi_-(z, \alpha), \quad (4.74 \alpha)$$

$$D(z, (\alpha)) = \text{diag} \left( (z - z_0)^{\kappa_1(\alpha)}, (z - z_0)^{\kappa_2(\alpha)}, \dots, (z - z_0)^{\kappa_k(\alpha)} \right) \quad (4.75 \alpha)$$

depend now on  $\alpha$ , and under the normalizing conditions (4.2):  $\Phi_+(\infty, \alpha) \equiv I$  and (4.76):  $\kappa_1(\alpha) \leq \kappa_2(\alpha) \leq \dots \leq \kappa_k(\alpha)$  are determined uniquely.

Assume now that for all  $\alpha$  from some neighborhood of some point  $\alpha_0 \in \mathbb{C}$  the condition  $\#(\mathcal{P}(\Phi(\alpha)) \cap G_+) = \#(\mathcal{N}(\Phi(\alpha)) \cap G_+)$  is satisfied (thus the block-entry  $S^r(\Phi(\alpha))_{11}$  is a square matrix for all  $\alpha$  which are close to  $\alpha_0$ ). Assume also that  $\det(S^r(\Phi(\alpha))_{11}) \neq 0$ , but  $\det(S^r(\Phi(\alpha_0))_{11}) = 0$ . According to Theorem 4.2, applied to the matrix-function  $\Phi(\alpha)$  with  $\alpha \neq \alpha_0$  (and  $\alpha$  which is close to  $\alpha_0$ ), the matrix function  $\Phi(z, \alpha)$  ( $\alpha \neq \alpha_0$ ) admits the factorization of the form  $\Phi(z, \alpha) = \Phi_+(z, \alpha) \Phi_-(z, \alpha)$ , in other words,  $D(z, \alpha) \equiv I$  for  $\alpha \neq \alpha_0$ . Moreover, the factors  $\Phi_+(z, \alpha)$ ,  $\Phi_-(z, \alpha)$  depend on  $\alpha$  holomorphically for  $\alpha \neq \alpha_0$  (this may be established from the explicit formulas (4.53), (4.54) for the factors). However, according to Theorem 4.1, the factorization of this form is impossible for  $\alpha = \alpha_0$  (otherwise the Lyapunov-Sylvester matrix  $S^r(\Phi(\alpha_0))$  would be invertible). Thus,  $D(\alpha_0) \neq I$ , i.e. not all  $\kappa_j(\alpha_0)$ ,  $j = 1, 2, \dots, k$ , vanish (although still  $\sum_{1 \leq j \leq k} \kappa_j(\alpha_0) = 0$ ). So, the factorization (4.74) undergoes a bifurcation at the value of the parameter  $\alpha$  such that  $\det S^r(\Phi(\alpha))_{11}$  vanishes. The natural question arise how to describe the bifurcation in a clear way? We believe that for rational matrix function  $\Phi$  in general position it may be done more or less explicitly.

## 5. SOME HISTORICAL REMARKS

There are many different ways to specify and represent analytic functions: for instance, Taylor series, decomposition in continuous fractions, representations by Cauchy integrals or by Fourier integrals, etc. (Of course, the distinction between different representation methods is often artificial and hard to make.) In the first half of 1970's this toolkit was enriched by an additional representation method: so called system realizations of analytic functions. The sources of the theory of system realizations belong to several different domains, in particular synthesis theory of linear electrical networks, the theory of linear control systems, and the theory of operator colligations (or nodes) and their characteristic functions. Investigations in these theories were carried out by representatives of different scientific disciplines. The investigations done by the mathematicians have their root in the pioneering work of M. S. Livšic, who is the forefather of the theory of system realizations.

In the middle of 1940's M. S. Livšic has introduced the notion of the characteristic function of a linear operator. This notion was first introduced for nonselfadjoint extensions of isometric operators with defect indices  $(1, 1)$  [L1] and more generally  $(n, n)$  [L2, L3] (for  $n > 1$  the characteristic function is matrix valued), and later for general operators with finite nonhermitian (or nonunitary) rank [L4]. M. S. Livšic has discovered the following properties of the characteristic function <sup>21</sup>.

1. The characteristic function determines the corresponding operator essentially uniquely up to unitary equivalence (first results of this kind are contained already in [L1]).
2. For each invariant subspace of the operator there is a decomposition of the characteristic function into a product of two factors: one of these factors is the characteristic function of the restriction of the original operator onto the given invariant subspace, and the other factor is the characteristic function of the compression of the original operator onto the corresponding coinvariant subspace.

Property 2 is the so called “multiplication theorem” for characteristic functions. Initially the multiplication theorem was established under various additional technical restrictions. The final formulation of the theorem has been obtained in the framework of the theory of operator colligations and their characteristic functions.

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<sup>21</sup>we formulate these properties, on purpose, in a somewhat rough and therefore slightly imprecise form

The theory of operator colligations that was created by M. S. Livšic (and that was further developed by his collaborator M. S. Brodskii, see the book [Br]) was a natural development of the theory of nonselfadjoint operators and their characteristic functions. This development was also intimately related with applications of the theory of commuting nonselfadjoint operators to physical problems, in particular to the problems of scattering and to elementary particles physics, and later to synthesis problems for electrical networks [L5, L6, LF]. The theory of open systems took a definitive shape in the works of M. S. Livšic in the early 1960's; the contents of these works were incorporated in the monograph [L9].

A theory parallel to the theory of open systems of M. S. Livšic has been developed by several other authors under the name of the theory of linear stationary dynamical systems. (For an exposition of system theory see [Z], [Fu], [KFA]; we especially recommend the survey [Kaas].) The transfer matrix function of such a system is precisely the characteristic function of the corresponding operator colligation. As M. S. Livšic has shown in several important examples, for scattering systems the characteristic function coincides with the scattering matrix [L5, L6].

A different line of investigation leading to the theory of system realizations is connected with control theory and with the name of R. Kalman (see [KFA]). Here one also encounters the notion of the transfer matrix function. Let us emphasize in this connection one important circumstance. In physical problems that M. S. Livšic was motivated by there appeared always an “energy balance” condition implying  $J$ -contractiveness of the characteristic (transfer) function and its “symmetry” with respect to the unit circle (or the real axis). It also imposed considering the adjoint operator  $A^*$  together with the state operator  $A$  of the system. Energy balance condition does not play an important role in control theory and this leads naturally to considering a general pair of operators  $A, B$  on the state space instead of the pair  $A, A^*$ . (In fact, R. Kalman develops system theory over arbitrary fields rather than over the field of complex numbers.)

Many results of the analytic theory of electrical networks can be considered as realization results for matrix functions of various classes. (There are many expositions of the theory of electrical networks; we recommend especially the monograph of V. Belevitch [Bel] that seems as if written for a mathematician. See also the survey [EfPo].) The well known Darlington's synthesis method for passive networks has been formulated for mathematicians by V. P. Potapov in 1966 as a realization problem for passive rational matrix functions [Pot3]; this problem was considered in details by Potapov's Ph. D. student E. Melamud [Me].

Already in [L4] M. S. Livšic used the theory of characteristic functions for the reduction of a nonselfadjoint operator to a triangular form, generalizing the theorem of I. Schur that an arbitrary matrix can be brought to an upper (or lower) triangular form by unitary equivalence. In this approach a multiplicative decomposition of the characteristic function corresponds to an “additive” decomposition of the operator itself over a linearly ordered chain of its invariant subspaces. This correspondence was used by M. S. Livšic in both directions. Using invariant subspaces of a finite-dimensional approximation of a given operator he decomposes the characteristic function of the approximating operator into factors, then passing to a limit he obtains a multiplicative decomposition of the characteristic function of the given operator, and finally using this multiplicative decomposition he constructs a triangular model of the given operator which is unitary equivalent to the operator itself.

A multiplicative decomposition of a meromorphic  $J$ -contractive matrix function on the unit disk (or on the upper half plane) has been obtained by V. P. Potapov in [Pot3] in a purely function theoretic way (the simpler case  $J = I$  was handled much earlier in [Pot1]). However an important special case of V. P. Potapov's theorem has already been obtained by M. S. Livšic in [L4] using operator theoretic methods; V. P. Potapov used the investigations of M. S. Livšic as



a guideline in his own function theoretic approach.

It is natural to ask whether one can reduce an operator to a diagonal form by a linear — no longer unitary — transformation. In the middle of 1950's M. S. Livšic has posed this problem to his then Ph. D. student L. A. Sakhnovich who has obtained numerous results in this direction. In particular the problem of the reduction of a nonselfadjoint operator to a diagonal form has lead L. A. Sakhnovich to a relation of the form

$$AX - XA^* = GJG^* \quad (5.1)$$

considered as an equation for  $X$  (formula (3) in [S1]). This relation appeared in a hidden form also in the theory of  $J$ -contractive matrix functions (see formula (22) and the following unnumbered formula in Chapter 2, Section 4 of [EfPo]) and in the theory of classical interpolation problems (formula (12) in Section 1 of [Kov]). A relation of the form (5.1) in [Kov] appears exactly in connection with what we called the chain identity; however the chain identity plays there a secondary role and is not highlighted. It should be mentioned that all the main ideas of the paper [Kov], published in 1983, have been suggested by V. P. Potapov some 10–12 years earlier; unfortunately, V. P. Potapov's contribution is not adequately reflected there. An identity of the form (5.1) has also been considered by L. de Branges in the framework of a certain generalized moment problem (in the language of Hilbert spaces of entire functions; see Theorem 27 of [Bran] and also formula (6.10) in [GolM]). A. A. Nudelman has used an identity of the form (5.1) at the basis of an abstract scheme that he developed for considering classical interpolation problems. It is interesting to note that an identity of the form (5.1) (and a related identity  $X - AXA^* = GJG^*$ ) appears in an entirely different context as well, namely in connection with fast inversion algorithms for structured matrices (Toeplitz, Hankel, Vandermonde, etc.); see the survey [KS].

Everything needed for the theory of system realizations in its present form has thus been available by the middle of 1970's. An important step was taken by L. A. Sakhnovich in [S2] (a detailed exposition of these results is contained in [S3]): he studied the spectral factorization of a rational matrix function  $R$  with both  $R$  and  $R^{-1}$  given as transfer functions of the corresponding linear systems (operator colligations). The spectra of the state space operators  $A$  and  $B$  of these systems do not intersect. One considers a pair of Sylvester–Lyapunov equations  $AT - TB = F_1G_1$ ,  $SA - BS = F_2G_2$ , where  $F_1$ ,  $G_1$ ,  $F_2$ ,  $G_2$  are the input and the output operators of the systems realizing  $R$ ,  $R^{-1}$ . It is shown that if the corresponding blocks of the solutions  $T$  and  $S$  of these equations are invertible then the matrix function  $R$  admits a spectral factorization, and formulas for the spectral factors (having the form of the formulas (4.53–4.54) of the present paper) are obtained. N. M. Kostenko (a Ph. D. student of L. A. Sakhnovich) has shown in [Kos] that the invertibility of these blocks is necessary for the existence of a spectral factorization. Relations which are analogous to zero-pole coupling relations were considered already in [S1] and used in [S2]. Let us notice that the spectral factors constructed in [S2] are simply the characteristic (transfer) matrix functions of subsystems arising by restricting the respective state space operators to the corresponding spectral subspaces. *This is the form taken by the multiplication theorem of M. S. Livšic in the current situation* (the matrices  $T$  and  $S$  define “metrics” which are now just bilinear functionals, neither positive definite nor even hermitian).

Unfortunately the paper [S2] did not have the impact it deserved. The subsequent development of the theory of system realizations is connected with the name of I. Gohberg. I. Gohberg has also lead and inspired a coherent work of many mathematicians in the theory of system realizations and its applications and this theory experienced a fast growth from the late 1970's onward. Already in 1979 there appeared the monograph [BGK1] dealing with spectral factorizations of rational matrix functions given as transfer functions of linear systems (operator

colligations). The paper [GKLR] considers the realization problem for matrix functions  $R$  and  $R^{-1}$  as transfer functions starting with “local data” (principal parts of Laurent series for  $R$  and  $R^{-1}$  at each pole); see also [BGR1]. In the beginning it was assumed that the pole sets for  $R$  and  $R^{-1}$  do not intersect; later the general case when these sets may intersect was considered as well. These and many other questions are considered in great details in the monograph [BGR2]; see also [KRR]. The collection of papers [CoMe] is dedicated to the spectral factorization for rational matrix functions based on the theory of system realizations; the papers [BGK2] and [BGK3] are especially related with our exposition in Section 4. There are results on system realization for rectangular (non-square) rational matrix functions given by local data [BGRa]. There are also realization results for matrix functions on a Riemann surface [BV], using deep new ideas of M. S. Livšic and his collaborators on characteristic functions for *commuting tuples* of nonselfadjoint operators [LKMV].

Factorization of matrix functions is a tool for many other problems, e.g., the theory of inverse problems for differential equations and prediction theory of stationary stochastic processes. If the corresponding matrix function is rational, this factorization (which is a technical tool for the original problem) may be carried out using system realizations which then become involved in the solution of the original problem as well. See, e.g., [AG1, AG2]. It is clear that the theory of system realizations can be successfully used also for the solution of the problems considered in [Yag]. It would be interesting to connect the questions considered in [Dei] with the theory of system realizations.

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